Jackson 11.4

Assume that a rocket ship leaves the Earth in the year 2000. One of a set of twins born in 1980 remains on Earth; the other rides in the rocket. The rocket ship is so constructed that it has an acceleration g in its own rest frame (this makes the occupants feel at home). It accelerates in a straight-line path for 5 years (by its own clocks), decelerates at the same rate for 5 more years, turns around, accelerates for 5 years, decelerates for 5 years, and lands on Earth. The twin in the rocket is 40 years old.

a) What year is it on Earth?

b) How far away from Earth did the rocket travel?

-added-

c) What was the maximum kinetic energy of the 50kg twin and for how long would an ordinary 1000 MW power plant have to operate to produce such an energy?

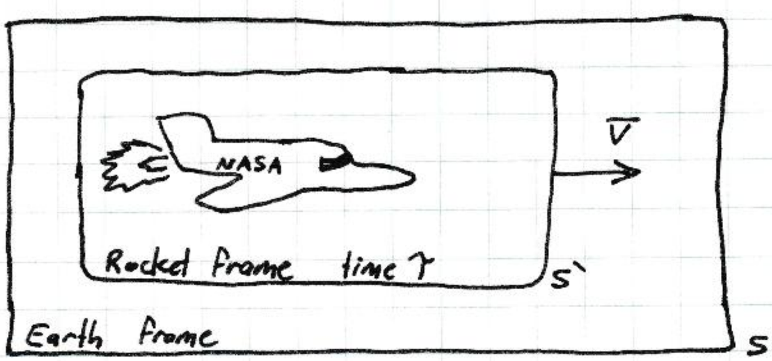
Don't look at SR for the whole answer – SR is a theory of inertial frames but since Newton's 1st Law does not hold true in the rocket frame, the rocket is non-inertial.

However, any frame will be inertial if you restrict yourself to a small enough region in spacetime. What is the appropriate region for this problem? I will assume that tidal gravity is not an issue here so the rocket frame can be spatially as large as we please. The problem is that the rocket is accelerating wrt a frame in which N1L does hold true (Earth).

So while the rocket may be initially inertial, it becomes non-inertial some short time later.

The obvious solution is to choose a rocket frame which is spatially large but temporally small. I will keep the rocket frame for an instant, after which it's no longer inertial so I'll choose another frame at a slightly higher speed wrt Earth in which the rocket is at rest.

Suppose we're at time $t = \tau$. The rocket has an acceleration g in its own rest frame, * however * it is momentarilly at rest. The whole rocket frame is moving at speed V wrt Terran observers.



The change in the rocket's speed wrt its own reference frame in some proper amount of differential time is:

$$dv` = a\, d\tau$$

After this proper time interval $d\tau$, the rocket's speed as measured by Earthly observers will be:

$$v + dv = \frac{dv' + v}{1 + v\,dv'} = \frac{g\,d\tau + v}{1 + g v\,d\tau} \qquad (g = a)$$

The denominator is of the form 1+ something small. We'll expand the denominator to order 1 in differentials.

$$v + dv = (g\,d\tau + v)(1 + g v\,d\tau)^{-1}$$

$$= (g\,d\tau + v)(1 - g v\,d\tau)$$

$$= g\,d\tau - \cancelto{0}{g^2 v\,d\tau^2} + v - g v^2\,d\tau$$

So the change in the rocket's speed wrt its own "rest" frame (and therefore its change in speed wrt Terra) is:

$$dv = g(1 - v^2)\,d\tau$$

$$\Longrightarrow \quad \frac{dv}{1 - v^2} = g\,d\tau$$

The rocket starts out at speed 0 at time 0 and reaches speed v at some time T.

$$\int_0^v \frac{dv}{1 - v^2} = g \int_0^T d\tau$$

The LHS integral can be solved by trig substitution, hyperbolic trig substitution or partial fractions. The speed as a function of time ought not be sinusoidal, so can the trig substitution. Partial fractions will yield logarithms — more plausible but still unlikely. The way to go is a hyperbolic trig substitution.

Let $V = \tanh\eta \quad dv = \operatorname{sech}^2\eta\, d\eta$

$$g\tau = \int_0^{\tanh^{-1}(v)} \frac{\operatorname{sech}^2\eta}{1-\tanh^2\eta}\, d\eta = \int_0^{\tanh^{-1}(v)} d\eta = \tanh^{-1}(v)$$

Therefore,

$$\boxed{V(\tau) = \tanh(g\tau)}$$

Rocket speed wrt Earth as a function of the rocket's own proper time

Since this is the rocket speed in Earth's frame, $V(\tau) = \frac{dx}{dt}$ as measured by Earth. We can use this function to find the rocket's distance from Earth as a function of proper rocket time via integration.

$$V = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{1}{\gamma}\frac{dx}{d\tau} = \sqrt{1-V^2(\tau)}\,\frac{dx}{d\tau}$$

Or,

$$\frac{dx}{d\tau}\sqrt{1-\tan^2(g\tau)} = V$$

$$\frac{dx}{d\tau}\operatorname{sech}(g\tau) = \tanh(g\tau)$$

$$\frac{dx}{d\tau} = \sinh(g\tau)$$

$$dx = \sinh(g\tau)\, d\tau$$

This is the differential equation relating the rocket's position from Earth as seen by Earth and the rocket's proper time. It is readily integrated

$$\int_0^x dx = \int_0^\tau \sinh(g\tau)\, d\tau$$

$$x = \frac{1}{g}\cosh(g\tau)\Big|_0^\tau$$

$$\boxed{x(\tau) = \frac{1}{g}\left[\cosh(g\tau) - 1\right]}$$

rocket position from Earth as seen by Earth as a function of rocket proper time.

With my choice of metric, the magnitude of all 4-velocities is -1:

$$u^\mu u_\mu = \left(\frac{dx}{d\tau}\right)^2 - \left(\frac{dt}{d\tau}\right)^2 = -1$$

Solving for $\frac{dt}{d\tau}$,

$$\frac{dt}{d\tau} = \sqrt{\sinh^2(g\tau) + 1} = \cosh(g\tau)$$

where I used the hyperbolic Pythagorean Theorem $\cosh^2 z - \sinh^2 z = 1$.

$$\boxed{\frac{dt}{d\tau} = \cosh(g\tau)}$$

Integrating both sides,

$$\boxed{t = \frac{1}{g}\sinh(g\tau)}$$

Relation between coordinate time and proper time.

Fascinating how acceleration yields an exponential relation between coordinate and proper time.

a) g in geometrical units:

$$9.81\,\mathrm{m/s^2}\,(3\times10^{8}\,\mathrm{m/s})^{-1} = 3.27\times10^{-8}\,/\mathrm{s}$$

The time (in everyone's frame) for the $\frac{1}{4}$ trip is $\frac{1}{4}$ of the time for the round trip.

$$t_{1/4} = \frac{1}{3.27\times10^{-8}/\mathrm{s}}\,\mathrm{SINH}\left[(3.27\times10^{-8}/\mathrm{s})(5\,\mathrm{yr})(3.15\times10^{7}\,\mathrm{sec/yr})\right]$$

$$= 2.64\times10^{9}\,\mathrm{sec}$$

$$= 83.7\,\mathrm{yr}$$

So the round trip as seen by Earth is:

$$t_{\substack{\text{round}\\\text{trip}}} = 4t_{1/4} = 335\,\mathrm{yr}$$

So if $T_{1/4} = 5\,\mathrm{yr}$, the rocket makes the round trip in 20 yrs proper time. The proper year is 2020. The rocket twin is 40 yrs old.

The round trip takes 335 yr coordinate time. The coordinate year is 2335. The Earth twin is 355 yrs old.

If, on the other hand, $T_{1/4} = 7\,\mathrm{yr}$,

$$t_{1/4} = \frac{1}{3.27\times10^{-8}/\mathrm{s}}\,\mathrm{SINH}\left[(3.27\times10^{-8}/\mathrm{s})(7\,\mathrm{yr})(3.15\times10^{7}\,\mathrm{sec/yr})\right]$$

$$= 2.07\times10^{10}\,\mathrm{sec}$$

$$= 657\,\mathrm{yr}$$

So the round trip as seen by Earth is:

$$t_{\substack{round\\trip}} = 4t_{1/4} = 2628\,yr$$

So if $T_{1/4} = 7yr$, the rocket makes the round trip in 28 years proper time. The proper year is 2028. The rocket twin is 48 yrs old.

The round trip takes 2628 yrs coordinate time. The coordinate year is 4628. The Earth twin is 2648 yrs old.

b) If $\tau_{1/4} = 5\,yrs$ the max distance the rocket travelled is

$$X_{max} = X(\tau_{1/2}) = 2X(\tau_{1/4}) = 2X(5yrs)$$

$$= \frac{2}{3.27\times10^{-8}/s}\left[\cosh\{(3.27\times10^{-8}/s)(5yrs)(3.15\times10^{7}sec/yr)\} - 1\right]$$

$$= 5.21\times10^{9}s\,(3\times10^{8}m/s)$$

$$= 1.56\times10^{18}\,m$$

$$= 165 \text{ light years}$$

If $\tau_{1/4} = 7\,yrs$, the max distance the rocket travelled is:

$$X_{max} = \frac{2}{3.27\times10^{-8}/s}(3\times10^{8}m/s)\left[\cosh[(3.27\times10^{-8}/s)(7yrs)(3.15\times10^{7}sec/yr)] - 1\right]$$

$$= 1.24\times10^{19}\,m$$

$$= 1310 \text{ light years}$$

Didn't quite make it to the centre of the galaxy...

c) First find V_{max} which for $\tau_{1/4}$ is 7 years. (proper time)

$$V_{max} = V(\tau = 7yr) = \tanh[(3.27\times10^{-8}/s)(7yrs)(3.15\times10^{7} sec/yr)]$$

$$= .999998908059$$

$$= 299999672 \ m/s$$

$$\Longrightarrow \quad \gamma = 677$$

So set $K_{e,max} \equiv P\Delta t$.

$$(\gamma-1)Mc^2 = P\Delta t \quad \Longrightarrow \quad \Delta t = \frac{(\gamma-1)Mc^2}{P}$$

$$\Delta t = \frac{(677-1)(50kg)(3\times10^{8} m/s)^2}{1000\times10^{6} J/s}$$

$$\boxed{\begin{aligned} \Delta t &= 3.0\times10^{12} sec \\ &= 9.7\times10^{4} yr \end{aligned}}$$

(!)

Time to read Miguel Alcubierre's paper on warp drive . . .

By this time we might even understand

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