Problem 7.14

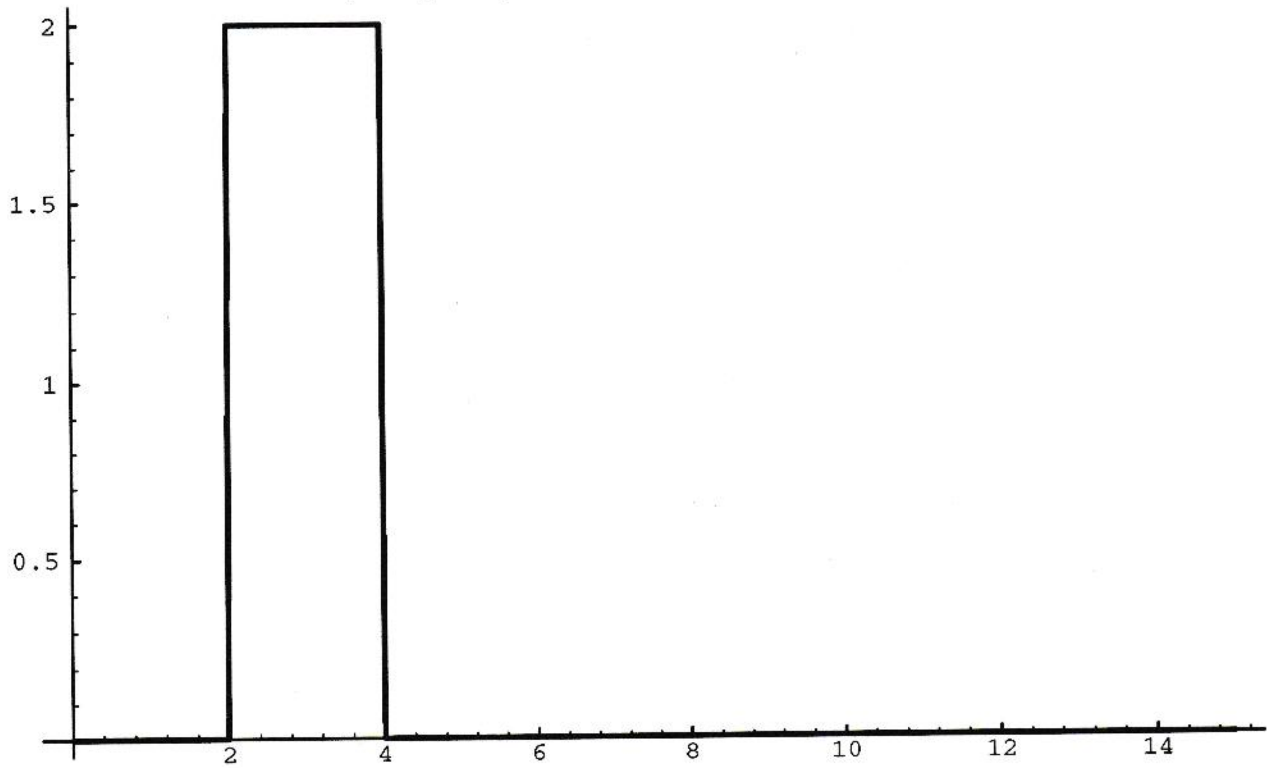
Use the Kramers - Kronig relation (7.120) to calculate the real part of $\epsilon(\omega)$, given the imaginary part of $\epsilon(\omega)$ for positive ω as

a) $\text{Im}\,\epsilon = \lambda\left[\theta(\omega-\omega_1) - \theta(\omega-\omega_2)\right] \qquad \omega_2 > \omega_1 > 0$

b) $\text{Im}\,\epsilon = \dfrac{\lambda\gamma\omega}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}$

In each case sketch the behavior of $\text{Im}\,\epsilon(\omega)$ and the result for $\text{Re}\,\epsilon(\omega)$ as functions of ω. Comment on the reasons for similarities or differences of your results as compared with the curves in Fig 7.8. The step function is $\theta(\bar{x}) = 0$, $\bar{x} < 0$ and $\theta(\bar{x}) = 1$, $\bar{x} > 0$

Part a (ima ginary) with lambda = 2, w_2 = 4, w_1 = 2



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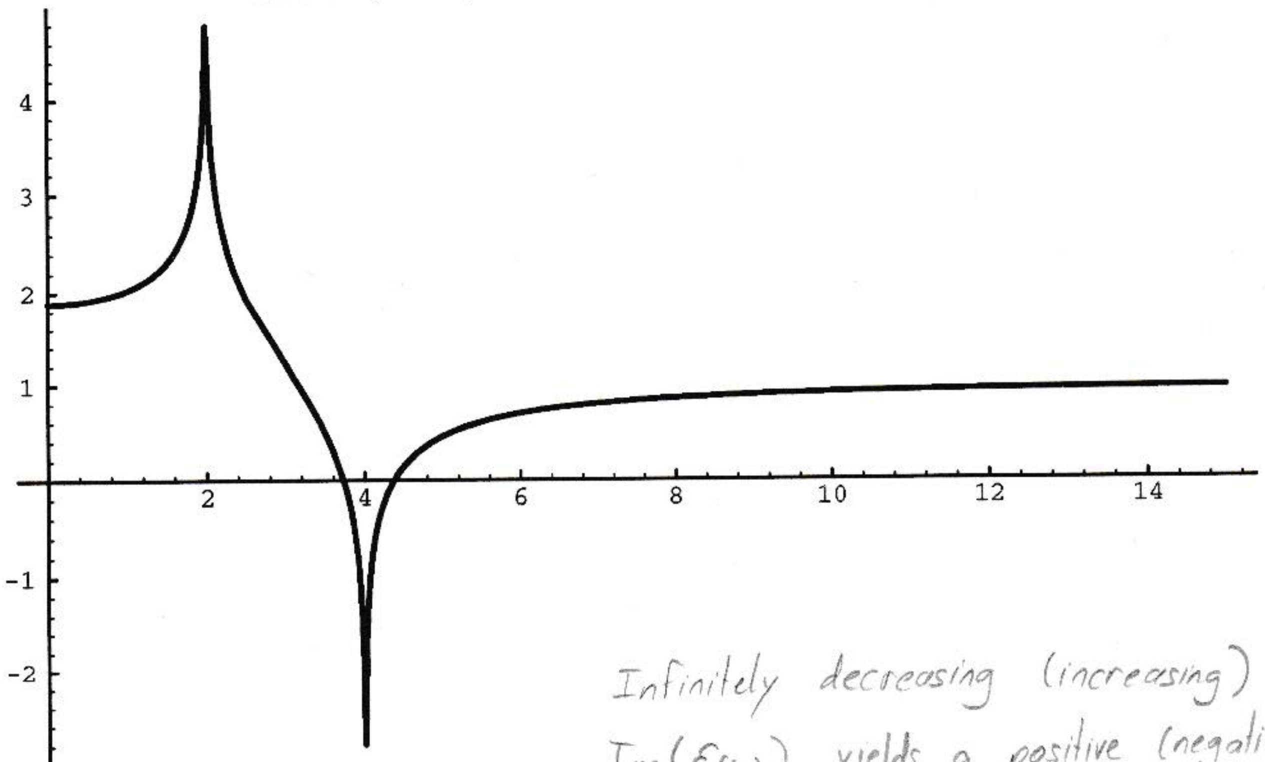


Infinitely decreasing (increasing) $Im(\varepsilon(\omega))$ yields a positive (negative) infinite $Re[\varepsilon(\omega)]$.

Of course, this is non-physical since nothing is truly infinite.

b)

$$\text{Re}[\varepsilon(\omega)] = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega'}{\omega'^2 - \omega^2} \frac{\lambda \gamma \omega'}{(\omega_0^2 - \omega'^2)^2 + \gamma^2 \omega'^2} \, d\omega'$$

$$= 1 + \frac{2\lambda\gamma}{\pi} P \int_0^\infty \frac{\omega'^2}{(\omega'^2 - \omega^2)\left[(\omega_0^2 - \omega'^2)^2 + \gamma^2 \omega'^2\right]} \, d\omega'$$

The integrand is even in ω' so I can extend the integral over the entire real line.

$$\text{Re}[\varepsilon(\omega)] = 1 + \frac{\lambda\gamma}{\pi} P \int_{-\infty}^\infty \frac{\omega'^2}{(\omega'^2 - \omega^2)\left[(\omega_0^2 - \omega'^2)^2 + \gamma^2 \omega'^2\right]} \, d\omega'$$

Now the poles need to be found. Clearly we have simple poles at $\omega' = \pm\omega$. We also have poles at:

$$(\omega_0^2 - \omega'^2)^2 + \gamma^2 \omega'^2 = 0$$

$$\omega_0^4 - 2\omega_0^2 \omega'^2 + \omega'^4 + \gamma^2 \omega'^2 = 0$$

$$\omega'^4 + [\gamma^2 - 2\omega_0^2]\omega'^2 + \omega_0^4 = 0$$

$$\omega'^2 = \frac{2\omega_0^2 - \gamma^2 \pm \sqrt{\gamma^4 - 4\omega_0^2\gamma^2 + 4\omega_0^4 - 4\omega_0^4}}{2}$$

$$= \omega_0^2 - \tfrac{1}{2}\gamma^2 \pm \gamma\sqrt{\gamma^2 - 4\omega_0^2}$$

We know that $\gamma \ll \omega_0$ from Jackson's section on frequency dependent dielectrics.

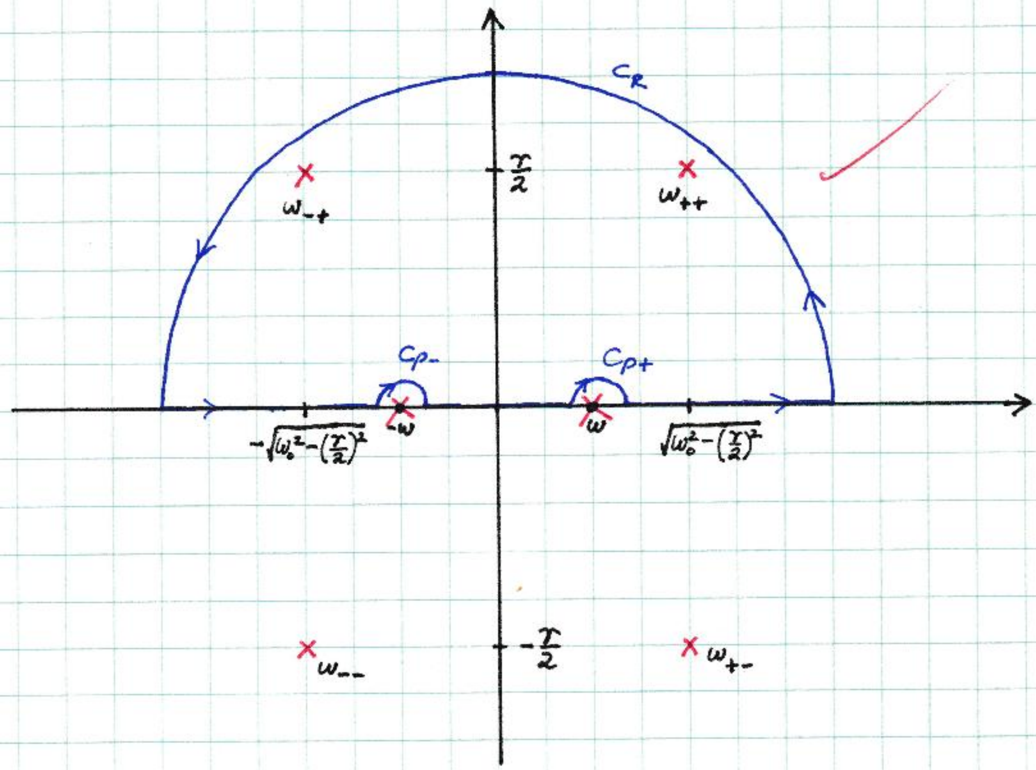
$$\omega'^2 = \omega_0^2 - \tfrac{1}{2}\gamma^2 \pm \frac{i\gamma}{2}\sqrt{4\omega_0^2 - \gamma^2}$$

$$\omega'^2 = \omega_0^2 - \tfrac{1}{2}\gamma^2 \pm i\gamma\sqrt{\omega_0^2 - \left(\tfrac{\gamma}{2}\right)^2}$$

$$= \left\{ \pm \frac{i\gamma}{2} + \sqrt{\omega_0^2 - \left(\tfrac{\gamma}{2}\right)^2} \right\}^2 \qquad \leftarrow \text{nifty}$$

So these 4 poles occur at:

$$\omega' = \pm \frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\tfrac{\gamma}{2}\right)^2}$$



Residues at $\omega' = \pm\omega$

The function to consider is $$\frac{\omega'}{2[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2]}$$

$$\text{Res}(\omega) = \frac{\omega}{2\left[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right]}$$

$$\text{Res}(-\omega) = -\frac{\omega}{2\left[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right]}$$

Residues at ω_{++} and ω_{-+}

The function to consider is:

$$\frac{\omega'^2}{(\omega'^2-\omega^2)\left[2(\omega_0^2-\omega'^2)(-2\omega')+2\gamma^2\omega'\right]} = \frac{\omega'}{2(\omega'^2-\omega^2)\left[2\omega'^2+\gamma^2-2\omega_0^2\right]}$$

$$\text{Res}(\omega_{++}) = \frac{\frac{i\gamma}{2}+\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}}{2\left(\omega_0^2-\frac{\gamma^2}{2}+i\gamma\sqrt{\omega_0^2-\frac{\gamma^2}{4}}-\omega^2\right)\left[2\omega_0^2-\gamma^2+2i\gamma\sqrt{\omega_0^2-\frac{\gamma^2}{4}}+\gamma^2-2\omega_0^2\right]}$$

$$= \frac{\frac{i\gamma}{2}+\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}}{4i\gamma\left(\omega_0^2-\omega^2-\frac{\gamma^2}{2}+i\gamma\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}\right)\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}}$$

$$\text{Res}(\omega_{-+}) = \frac{\frac{i\gamma}{2}-\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}}{2\left(\omega_0^2-\frac{\gamma^2}{2}-i\gamma\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}-\omega^2\right)\left[2\omega_0^2-\gamma^2-2i\gamma\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}+\gamma^2-2\omega_0^2\right]}$$

$$= \frac{\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}-\frac{i\gamma}{2}}{4i\gamma\left(\omega_0^2-\omega^2-\frac{\gamma^2}{2}-i\gamma\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}\right)\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}}$$

Now I'll perform the integral

$$I_c = \int_c \frac{\omega'^2}{(\omega'^2-\omega^2)\left[(\omega_0^2-\omega'^2)^2+\gamma^2\omega'^2\right]}\,d\omega'$$

where $C = C_R + C_{p+} + C_{p-} + P$ is drawn on page 7.14-4.

By some nameless theorem on integrals over piecewise continuous paths:

$$I_c = I_{C_R} + I_{C_{p+}} + I_{C_{p-}} + PI$$

where "P" denotes "principle value" on the real axis. In appendix A (last page of this problem) I prove that $I_{C_R} = 0$.

By Dr. Singh's residue theorem:

$$I_{C_{p+}} + I_{C_{p-}} = -\pi i\, Res(\omega) - \pi i\, Res(-\omega)$$

But on pg 7.14-5 I found that $Res(\omega) = -Res(-\omega)$, so:

$$I_{C_{p+}} + I_{C_{p-}} = 0.$$

Therefore, we have:

$$I_c = PI$$

where PI is really what we're after.

By the residue theorem,

$$I_c = 2\pi i \, Res(\omega_{++}) + 2\pi i \, Res(\omega_{-+})$$

$$= \frac{\pi}{2\gamma\sqrt{\omega_0^2-(\frac{\gamma}{2})^2}}\left[\frac{\sqrt{\omega_0^2-(\frac{\gamma}{2})^2} + \frac{i\gamma}{2}}{\left(\omega_0^2-\omega^2-\frac{\gamma^2}{2}+i\gamma\sqrt{\omega_0^2-(\frac{\gamma}{2})^2}\right)} + \frac{\sqrt{\omega_0^2-(\frac{\gamma}{2})^2} - \frac{i\gamma}{2}}{\omega_0^2-\omega^2-\frac{\gamma^2}{2}-i\gamma\sqrt{\omega_0^2-(\frac{\gamma}{2})^2}}\right]$$

Let $\psi = \omega_0^2 - \omega^2 - \frac{\gamma^2}{2}$ and $\phi = \sqrt{\omega_0^2-(\frac{\gamma}{2})^2}$.

$$I_c = \frac{\pi}{2\gamma\phi}\left[\frac{\phi+\frac{i\gamma}{2}}{\psi+i\gamma\phi} + \frac{\phi-\frac{i\gamma}{2}}{\psi-i\gamma\phi}\right]$$

$$= \frac{\pi}{2\gamma\phi}\left[\frac{(\phi+\frac{i\gamma}{2})(\psi-i\gamma\phi)}{\psi^2+\gamma^2\phi^2} + \frac{(\phi-\frac{i\gamma}{2})(\psi+i\gamma\phi)}{\psi^2+\gamma^2\phi^2}\right]$$

$$= \frac{\pi}{2\gamma\phi}\left[\frac{\phi\psi - i\gamma\phi^2 + \frac{1}{2}i\gamma\psi + \frac{1}{2}\gamma^2\phi + \phi\psi + i\gamma\phi^2 - \frac{1}{2}i\gamma\psi + \frac{1}{2}\gamma^2\phi}{\psi^2+\gamma^2\phi^2}\right]$$

$$= \frac{\pi}{2\gamma\phi}\left[\frac{2\phi\psi + \gamma^2\phi}{\psi^2+\gamma^2\phi^2}\right]$$

$$= \frac{\pi}{2\gamma}\left[\frac{2\psi + \gamma^2}{\psi^2+\gamma^2\phi^2}\right]$$

$$= \frac{\pi}{2\gamma}\left[\frac{2(\omega_0^2-\omega^2-\frac{\gamma^2}{2}) + \gamma^2}{(\omega_0^2-\omega^2-\frac{\gamma^2}{2})^2 + \gamma^2(\omega_0^2-\frac{\gamma^2}{4})}\right]$$

$$= \frac{\pi}{2\gamma}\left[\frac{2\omega_0^2 - 2\omega^2}{\omega_0^4+\omega^4+\cancel{\frac{\gamma^4}{4}}-2\omega_0^2\omega^2-\cancel{\omega_0^2\gamma^2}+\omega^2\gamma^2+\cancel{\gamma^2\omega_0^2}-\cancel{\frac{\gamma^4}{4}}}\right]$$

$$I_c = \frac{\pi}{\gamma}\left[\frac{\omega_0^2 - \omega^2}{\omega_0^4 + \omega^4 - 2\omega_0^2\omega^2 + \omega^2\gamma^2}\right]$$

$$= \frac{\pi}{\gamma}\left(\frac{\omega_0^2 - \omega^2}{[\omega_0^2 - \omega^2]^2 + \omega^2\gamma^2}\right)$$

And as found on pg 7.14-6,

$$P\int_{-\infty}^{\infty} \frac{\omega'^2}{(\omega'^2 - \omega^2)[(\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2]}\, d\omega' \quad = \quad I_c$$

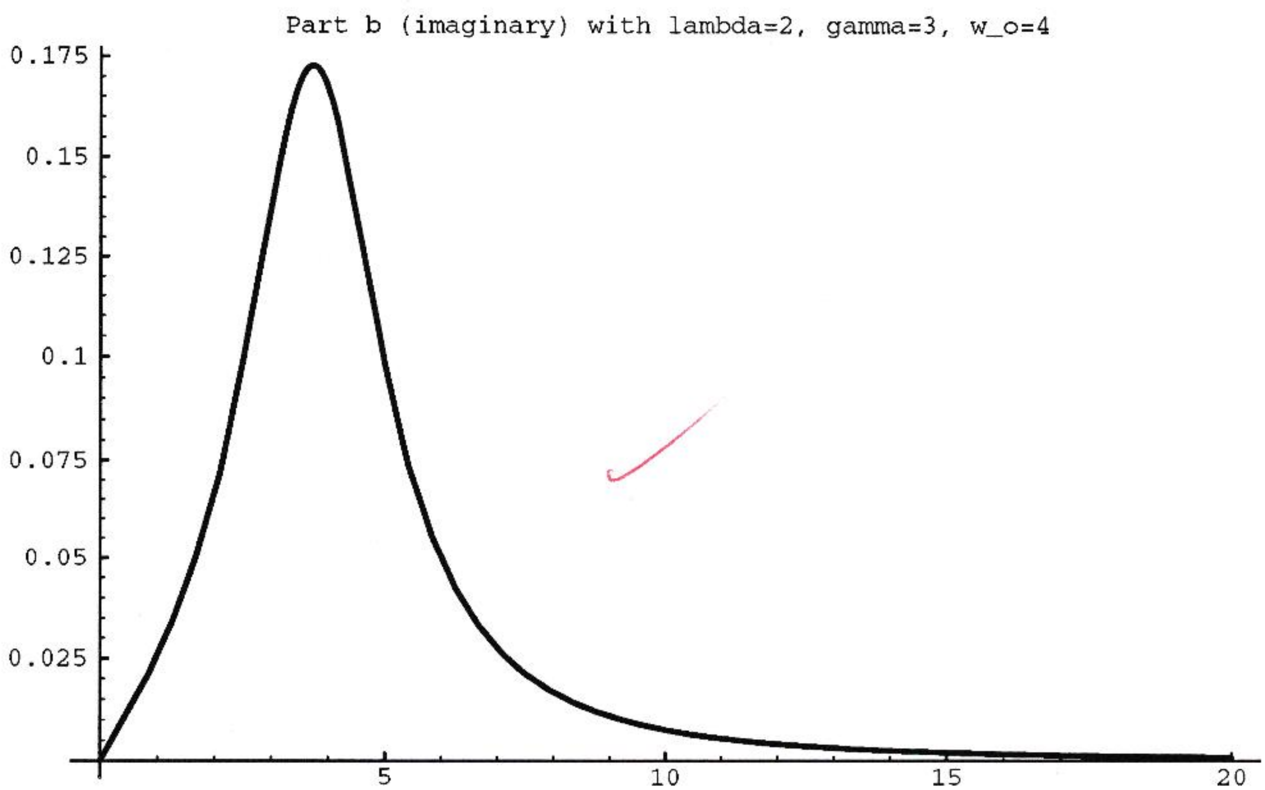
$$= \frac{\pi}{\gamma}\left(\frac{\omega_0^2 - \omega^2}{[\omega_0^2 - \omega^2]^2 + \omega^2\gamma^2}\right)$$

So, going back to pg 7.14-3

$$Re[\varepsilon(\omega)] = 1 + \frac{\lambda\gamma}{\pi} P\int_{-\infty}^{\infty} \frac{\omega'^2}{(\omega'^2 - \omega^2)[(\omega_0^2 - \omega'^2)^2 + \gamma^2\omega'^2]}\, d\omega'$$

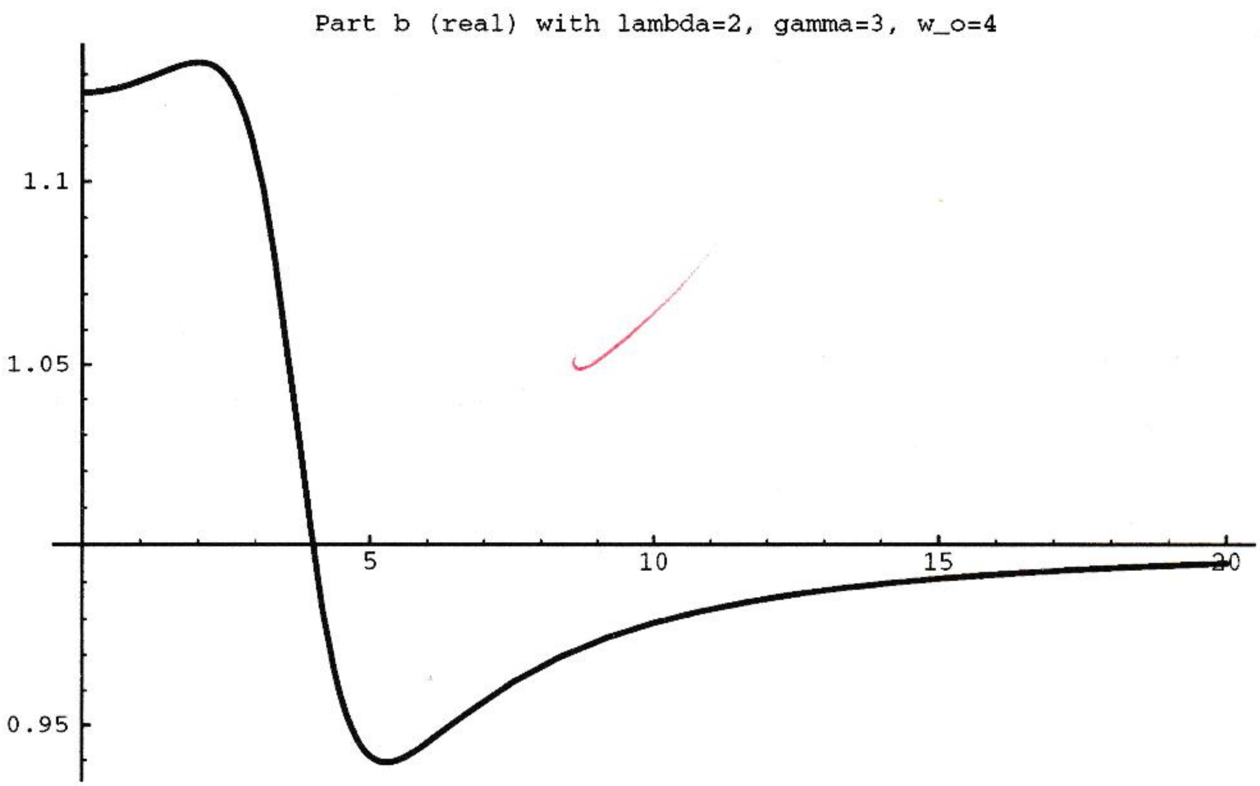
$$= 1 + \frac{\lambda\gamma}{\pi}\,\frac{\pi}{\gamma}\left(\frac{\omega_0^2 - \omega^2}{[\omega_0^2 - \omega^2]^2 + \omega^2\gamma^2}\right)$$

$$= 1 + \lambda\left[\frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}\right]$$



Now this is like Jackson's graph. The only difference is that he has 2 resonances ~~when~~ while we have only 1.

Note $Im[\varepsilon(\omega)]$ is large when $Re[\varepsilon(\omega)]$ is small, as expected.


Part b (real) with lambda=2, gamma=3, w_o=4
1.1
1.05
0.95
5
10
15
20