Problem 7.8

A stylized model of the ionosphere is a medium described by the dielectric constant (7.59). Consider the Earth with such a medium beginning suddenly at a height h and extending to infinity. For waves with polarization both perpendicular to the plane of incidence (from a horizontal antenna) and in the plane of incidence (from a vetical antenna),

a) Show from Fresnel's equations for reflection and refraction that for $\omega > \omega_p$ there is a range of angles of incidence for which reflection is not total, but for larger angles there is total reflection back to Earth.

b) A radio amateur operating at a wavelength of 21 meters in the early evening finds that she can receive distant stations located more than 1000 km away, but none closer. Assuming that the signals are being reflected from the F layer of the ionosphere at an effective height of 300 km, calculate the electron density. Compare with the known maximum and minimum F layer densities of $\sim 2 \times 10^6\ cm^{-3}$ in the daytime and $\sim (2\text{-}4) \times 10^5\ cm^{-3}$ at night.

max: 35
me: 35
avg: 30
σ: 5

a) First off, we must ask ourselves: why do we have the restriction that $\omega > \omega_p$? Consider the wave number of a plane wave travelling through a plasma:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2} \qquad \text{(Jackson 7.61)}$$

which is the dispersion relation for $\omega = \omega(k)$ for plasma frequencies. If $\omega < \omega_p$, the wave number is complex; the oscillatory part of the plane wave has real and imaginary parts

$$e^{i(\bar{k}\cdot\bar{x} - \omega t)} = \underbrace{e^{-|k|x}}_{\text{exponential decay}} \; \underbrace{e^{-i\omega t}}_{\text{oscillatory}}$$

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which is not a travelling wave — it's a standing wave which oscillates in time with an amplitude that decreases exponentially with distance. No energy is propagated and the wave dies quickly.

Therefore, plasmas act as a hi-pass filter with the cutoff frequency being ω_p, the plasma frequency. If we are to consider anything, we must consider plane wave frequencies greater than the plasma frequency, $\omega > \omega_p$.

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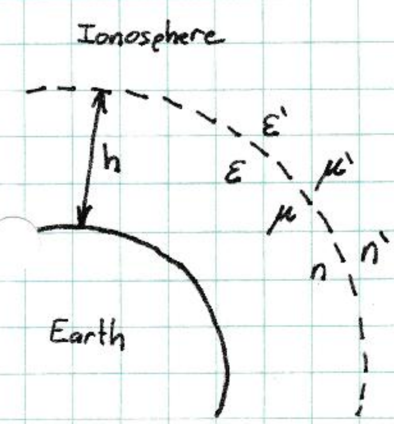
As with most non-ferromagnetic medium,

$$\mu = \mu' = 1$$



Since the region under the ionosphere is air, we'll take

$$\varepsilon = 1 \,, \qquad n = 1 \,.$$

And now, obtaining an expression for n',

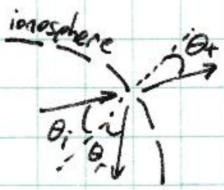
$$n_{ionosphere} = n' = \frac{c\,k_{plasma}}{\omega} = \frac{c}{\omega}\frac{\sqrt{\omega^2-\omega_p^2}}{c}$$

$$\Longrightarrow \quad n' = \sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2} \quad \text{— give eq 7.59}$$

Since $n' = \sqrt{\mu'\varepsilon'}$, we also have:

$$\varepsilon' = 1-\left(\frac{\omega_p}{\omega}\right)^2$$

As usual, t = transmitted, r = reflected, i = incident. θ_i will be the angle of incidence, which incidently, equals θ_r, the angle of reflection. θ_t is the angle of refraction.



From Snell's Law, $n_i \sin\theta_i = n_t \sin\theta_t$

$$\Longrightarrow \quad \sin\theta_t = \frac{n_i}{n_t}\sin\theta_i = \frac{\sin\theta_i}{\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}$$

We run into trouble when

$$\sin\theta_t = \frac{\sin\theta_i}{\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}} > 1 \quad \Longrightarrow \quad \sin\theta_i > \sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$$

Define a critical angle for any given frequency ω (Jackson calls this i_0)

$$\theta_c = \sin^{-1}\left(\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}\right)$$

So Snell's Law can be restated as:

$$\sin\theta_t = \frac{\sin\theta_i}{\sin\theta_c}$$

So what happens when

$$\sin\theta_t = \frac{\sin\theta_i}{\sin\theta_c} > 1 \quad ?$$

The angle θ_t in real life remains at $\frac{\pi}{2}$ but now $\sin\theta_t > 1$. As Jackson points out on pg 283, if $\sin\theta_t > 1$ then θ_t is a complex angle with a purely imaginary cosine. When this happens,

$$\cos\theta_t = i\sqrt{\sin^2\theta_t - 1} = i\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}$$

why does it have to be pure imaginary?

The problem asks us to show that there are a range of angles θ_i for which there is total reflection, using the Fresnel equations. It's no great surprise that there are angles which there is both reflected and refracted waves — that's the way things normally behave! (and is easily seen from the Fresnel equations).

What is more difficult to see, and the intent of the problem, is that there is a range of angles which have total reflection of waves. This fact is not immediately obvious from looking at Jackson (7.39) and (7.41), the Fresnel equations.

I'll start off with $\vec{E}$ // to the plane of incidence.

start here!

$$R = \left|\frac{E_r}{E_i}\right| = \frac{n_t^2\cos\theta_i - \sqrt{n_t^2 - \sin^2\theta_i}}{n_t^2\cos\theta_i + \sqrt{n_t^2 - \sin^2\theta_i}} = \frac{n_t^2\cos\theta_i - n_t\cos\theta_r}{n_t^2\cos\theta_i + n_t\cos\theta_r}$$

As I mentioned earlier, if $\theta_i > \sin^{-1}\left(\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}\right)$

$$\frac{E_r}{E_i} = \frac{n_t^2\cos\theta_i - n_t i\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}{n_t^2\cos\theta_i + n_t i\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}$$

$$= \frac{n_t\cos\theta_i - i\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}{n_t\cos\theta_i + i\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}$$

This expression is of the form:

$$\frac{E_r}{E_i} = \frac{A - iB}{A + iB} = \frac{\sqrt{A^2+B^2}\, e^{-i\phi}}{\sqrt{A^2+B^2}\, e^{i\phi}} = e^{-2i\phi} \qquad \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

$R = |E_r/E_i| = 1$ — simple.

Thus, the reflected wave has a phase shift from the incident wave, but since $|e^{-2i\phi}| = 1$, $|E_r| = |E_i|$. All the energy of the incident wave goes into the reflected wave — there's no energy left for a transmitted (refracted) wave.

So when $\sin\theta_i > \sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$ and $\bar{E}$ is $\parallel$ to the plane of incidence, there's total internal reflection.

I'll repeat this for $\bar{E}$ $\perp$ to the plane of incidence. From Jackson (7.39) pg 281,

$$\frac{E_r}{E_i} = \frac{\cos\theta_i - \sqrt{n_+^2 - \sin^2\theta_i}}{\cos\theta_i + \sqrt{n_+^2 - \sin^2\theta_i}} = \frac{\cos\theta_i - n_+\cos\theta_r}{\cos\theta_i + n_+\cos\theta_r}$$

$$= \frac{\cos\theta_i - i n_+\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}}{\cos\theta_i + i n_+\sqrt{\left(\frac{\sin\theta_i}{\sin\theta_c}\right)^2 - 1}} \qquad \left(\text{when } \theta_i > \sin^{-1}\left(\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}\right)\right)$$

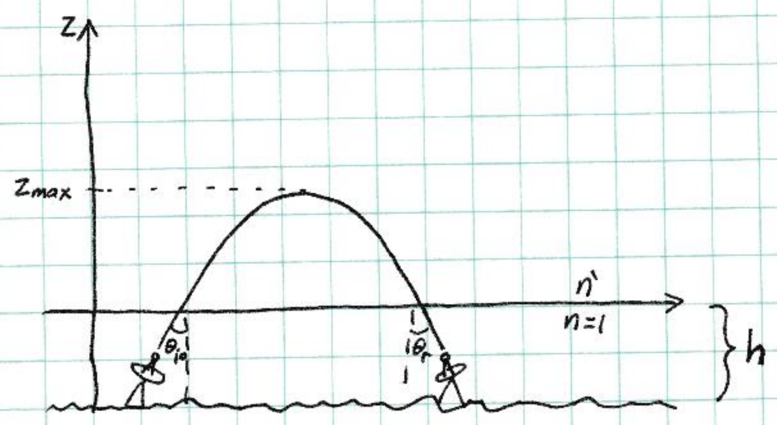
Which is of the form

$$\frac{E_r}{E_i} = \frac{A - iB}{A + iB} = \frac{\sqrt{A^2+B^2}\, e^{-i\phi}}{\sqrt{A^2+B^2}\, e^{i\phi}} = e^{-2i\phi}$$

where $\phi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{n_+\cos\theta_r}{\cos\theta_i}\right)$ as before.

As in the previous case, all the incident energy gets reflected — E_r and E_i differ only by a phase.

b)



From before we had $n' = \sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$ and $n=1$, as shown above.

From Jackson (7.60) pg 288 we have:

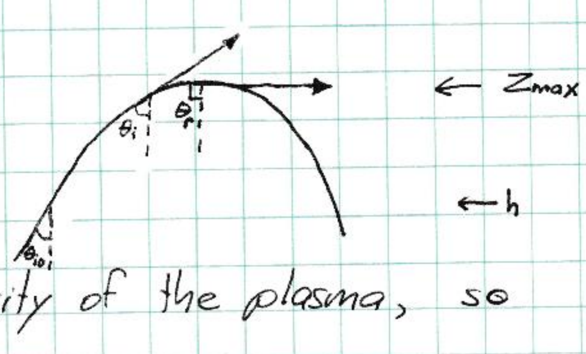
$$n' = \sqrt{1 - \frac{4\pi N Z e^2}{m\omega^2}} \qquad (i)$$

Since $N = N(z)$ is a function of z, $n' = n'(z)$ is a function of z. That is, the index of refraction changes with height. Moreover, for each point along the wave's path, Snell's Law gives: note, θ_r is a function of z as is θ_i but θ_{io} is the beginning angle.

$$n \sin\theta_i = n' \sin\theta_r \implies \sin\theta_{io} = n'(z)\sin\theta_r(z) = \text{constant}$$

At the top of the wave's trajectory where $z = z_{max}$, $\theta_r = \frac{\pi}{2}$ (the wave is refracted horizontally), which means

$$\sin\theta_{io} = n'(z_{max}) = \sqrt{1 - \frac{4\pi N(z_{max}) Z e^2}{m\omega^2}}$$



Jackson says that NZ is the density of the plasma, so $N(z_{max})Z = \rho(z_{max})$

This value falls between the daytime and nighttime values. The problem did