very nice job

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Analytic Mechanics
Problem # 9
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Air resistance force acting on a particle is given by $\vec{F}_{drag} = -CV^2\hat{V}$, where V is the particle's speed wrt the air, and $\hat{V}$ is a unit vector in the direction of its velocity. The particle is launched upward with an initial speed V_0, starting from ground level $(y=0)$ at the clock reading $t=0$.

a) Outline solutions for the particle's velocity as a function of time both when rising and subsequently when falling back towards the ground.

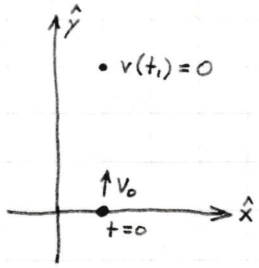
b) Find expressions for the maximum height reached by the particle and the time required to get there.

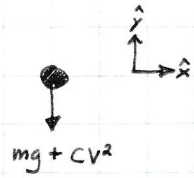
c) Expand your results in a series of powers of C assuming that C is small and keeping enough terms to find first order correction to standard results in absence of air resistance.

d) Use the results of b as an initial condition for the particle's return to the ground. Find the particle's speed and the clock reading at impact.

part a

Draw pictures + label stuff. Draw FBD





note: drag force acts opposite to motion. I am considering $\vec{v}(t) = v(t)\hat{y}$ only for $v(t) > 0$ (upwards motion). Therefore, drag acts downward.

Apply N2L to FBD.

$$\sum F_y = -mg - cv^2 = ma_y \quad \Rightarrow \quad a_y = -\left[g + \frac{c}{m}v^2\right]$$

① Since drag acts opposite to motion, the above expression for a_y is valid only for the particle going up; coming down, drag will act upwards (and be of opposite sign of mg).

② The above equation is a non-linear 1st order differential equation in v. It begs to be separated.

$$\frac{dv}{dt} = -\left[g + Dv^2\right] \qquad D \overset{def}{\equiv} \frac{c}{m}$$

$$\frac{dv}{Dv^2 + g} = -dt$$

$$\int_{v=v(0)}^{v=v(t^*)} \frac{dv}{Dv^2 + g} = -\int_{t=0}^{t=t^*} dt$$

substitution: $v = \sqrt{\frac{g}{D}}\, TAN\theta$

$$dv = \sqrt{\frac{g}{D}}\, SEC^2\theta\, d\theta$$

$$\theta = TAN^{-1}\left(\sqrt{\frac{D}{g}}\, v\right) \checkmark$$

$$\sqrt{\frac{g}{D}} \int_{v=v(0)}^{v=v(t^*)} \frac{SEC^2\theta}{g\,SEC^2\theta}\, d\theta = -t^*$$

$$\frac{1}{\sqrt{gD}}\, TAN^{-1}\left(\sqrt{\frac{D}{g}}\, v\right)\Bigg|_{v=v(0)}^{v=v(t^*)} = -t^* \checkmark$$

$$TAN^{-1}\left[\sqrt{\frac{D}{g}}\, v(t^*)\right] - TAN^{-1}\left[\sqrt{\frac{D}{g}}\, V_0\right] = -t^*\sqrt{gD} \checkmark$$

aka $\eta = Tan(\eta_0 - \tau)$

Probably most efficient to use the form

$$\boxed{v(t) = \sqrt{g/D}\, Tan\left[Tan^{-1}\sqrt{\frac{D}{g}}\, V_0 - \sqrt{gD}\, t\right]}$$

No need for dummy variables anymore so the * will be dropped. Taking TANGENT of both sides

$$TAN\left[TAN^{-1}\left(\sqrt{\frac{D}{g}}\, v(t)\right) - TAN\left(\sqrt{\frac{D}{g}}\, V_0\right)\right] = TAN\left(-t\sqrt{gD}\right) \quad (i)$$

Any good student of physics should know: (instant recall) ☺

① $TAN(\psi \pm \phi) = \dfrac{TAN\psi \pm TAN\phi}{1 \mp TAN\psi\, TAN\phi}$ ✓

② $TAN(-\eta) = -TAN\eta$

③ $TAN\,TAN^{-1}\Phi = \Phi$ ✓

Thus, eq(1) becomes:

$$\frac{\sqrt{\frac{D}{g}}\,v(t) - \sqrt{\frac{D}{g}}\,V_0}{1 + \frac{D}{g}V_0 V(t)} = -\,TAN\left(t\sqrt{gD}\right)$$

$$A\,V(t) = A V_0 - TAN\left(t\sqrt{gD}\right) - A^2 V_0\, V(t)\, TAN\left(t\sqrt{gD}\right) \qquad \boxed{A^2 \equiv \frac{D}{g}}$$

$$v(t)\left[A + A^2 V_0\, TAN\left(t\sqrt{gD}\right)\right] = A V_0 - TAN\left(t\sqrt{gD}\right)$$

$$v(t) = \frac{A V_0 - TAN\left(t\sqrt{gD}\right)}{A + A^2 V_0\, TAN\left(t\sqrt{gD}\right)}$$

$$= \frac{\sqrt{\frac{D}{g}}\,V_0 - TAN\left(t\sqrt{gD}\right)}{\sqrt{\frac{D}{g}} + \frac{D}{g}V_0\, TAN\left(t\sqrt{gD}\right)}$$

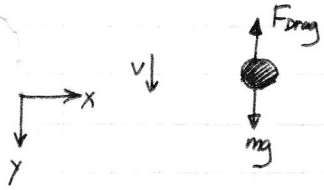
$$= \frac{\sqrt{\frac{c}{mg}}\,V_0 - TAN\left(t\sqrt{\frac{gc}{m}}\right)}{\sqrt{\frac{c}{mg}} + \frac{c}{mg}V_0\, TAN\left(t\sqrt{\frac{gc}{m}}\right)}$$

$$v(t) = \frac{V_0 - \sqrt{\frac{mg}{c}}\, TAN\left(t\sqrt{\frac{gc}{m}}\right)}{1 + \sqrt{\frac{c}{mg}}\,V_0\, TAN\left(t\sqrt{\frac{gc}{m}}\right)} \quad ✓$$

particle's speed on the way up.

Since $\tan(0)=0$ we get the correct $v(t=0_s)$.

Now we let the particle fall from rest.



Applying N2L: $mg - F_{Drag} = ma_y$

$$mg - CV^2 = ma_y$$

Solving for a_y,

$$a_y = g - \frac{c}{m}V^2$$

I have assumed that F_{drag} acts opposite in direction to $m\bar{g}$. Since F_{drag} acts in the $-\hat{v}$ direction this equation is valid only if the particle is falling.

Again, the same technique can be applied to solve this equation with one difference.

$$\frac{dv}{dt} = g - \frac{c}{m}V^2$$

$$\frac{dv}{g - DV^2} = dt \qquad D \equiv \frac{c}{m}$$

$$\int_{V=V(0)}^{V=V(t)} \frac{dv}{g - DV^2} = \int_{t=0}^{t^*=t} dt^*$$

$$V = \sqrt{\frac{g}{D}}\,\text{TANH}\,\theta$$

$$dv = \sqrt{\frac{g}{D}}\,\text{SECH}^2\theta\, d\theta$$

$$\sqrt{\frac{g}{D}} \int_{V=V(0)}^{V=V(t)} \frac{\text{SECH}^2\theta}{g(1 - \text{TANH}^2\theta)} = \frac{1}{\sqrt{gD}} \int_{V=V(0)}^{V=V(t)} d\theta = \frac{1}{\sqrt{gD}}\,\text{TANH}^{-1}\left(V\sqrt{\frac{D}{g}}\right)\Bigg|_{V=V(0)}^{V=V(t)}$$

$$\sqrt{\frac{m}{gc}}\ \tanh^{-1}\left(V\sqrt{\frac{c}{mg}}\right)\Bigg|_{V=V(0)}^{V=V(t)}$$

Since the particle falls from rest $v(0) = 0\,m/s$

$$\sqrt{\frac{m}{gc}}\ \tanh^{-1}\left[v(t)\sqrt{\frac{c}{mg}}\right] = \int_0^t dt^* = t$$

Solving for $v(t)$, which I'll call $V_{down}(t)$:

$$V_{down}(t) = \sqrt{\frac{mg}{c}}\ \tanh\left(t\sqrt{\frac{gc}{m}}\right)$$ ✓ good

Note that my answer differs from your answer by a minus sign. This is to be expected because I defined up* differently than you. a

Actually, our solutions are in agreement. good

* "positive" would be a better word here. yup.

Part b

Start with the result from part a:

$$\eta = \mathrm{TAN}[\Psi_0 - \tau]$$

From the way we defined η,

$$v(t) = V_T \, \mathrm{TAN}[\Psi_0 - \tau]$$

$$\int_0^t v(t)\,dt = V_T \int_0^t \mathrm{TAN}[\Psi_0 - \tau]\,dt$$

Now the variable of integration needs to be matched with the variable of the integrand. Recall:

$$\tau = \frac{gt}{V_T} \Rightarrow t = \frac{V_T \tau}{g} \Rightarrow dt = \frac{V_T}{g}\,d\tau$$

$$y(t) = \int_0^t v(t)dt = \frac{V_T^2}{g} \int_0^{V_T t/g} \mathrm{TAN}[\Psi_0 - \tau]\,d\tau$$

$$= -\frac{V_T^2}{g} \ln\left|\mathrm{SEC}(\Psi_0 - \tau)\right| \Bigg|_{\tau=0}^{\tau=\frac{V_T t}{g}}$$

$$= \frac{V_T^2}{g} \ln\left|\cos(\Psi_0 - \tau)\right| \Bigg|_{\tau=0}^{\tau=\frac{V_T t}{g}}$$

$$= \frac{V_T^2}{g}\left\{ \ln\left|\cos\left(\Psi_0 - \frac{gt}{V_T}\right)\right| - \ln\left|\cos \Psi_0\right| \right\}$$

$$= \frac{V_T^2}{g} \ln\left| \frac{\cos\left(\Psi_0 - \frac{gt}{V_T}\right)}{\cos \Psi_0} \right|$$

$$y(t) = \frac{V_T^2}{g} \ln \left| \frac{\cos\left[\tan^{-1}(M_0) - \frac{gt}{V_T}\right]}{\cos\tan^{-1} M_0} \right|$$

$$= \frac{V_T^2}{g} \ln \left| \frac{\cos\left[\tan^{-1}\left(\frac{V_0}{V_T}\right) - t\sqrt{\frac{gc}{m}}\right]}{\cos\tan^{-1}\left(\frac{V_0}{V_T}\right)} \right|$$

Finally,

$$y(t) = \frac{m}{c} \ln \left| \frac{\cos\left[\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right) - t\sqrt{\frac{gc}{m}}\right]}{\cos\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)} \right| \quad \checkmark$$

Let the time in which the ball comes to rest be t_f. This is the time at which $v(t) = 0\,m/s$ when $t = t_f$.

$$v(t) = \frac{V_0 - \sqrt{\frac{mg}{c}} \tan\left(t\sqrt{\frac{gc}{m}}\right)}{1 + \sqrt{\frac{c}{mg}} V_0 \tan\left(t\sqrt{\frac{gc}{m}}\right)} \overset{set}{\equiv} 0$$

$$\Longrightarrow \quad V_0 - \sqrt{\frac{mg}{c}} \tan\left(t_f\sqrt{\frac{gc}{m}}\right) = 0$$

$$\tan\left(t_f\sqrt{\frac{gc}{m}}\right) = V_0\sqrt{\frac{c}{mg}}$$

$$t_f\sqrt{\frac{gc}{m}} = \tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)$$

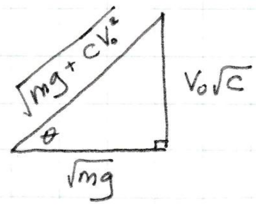
$$t_f = \sqrt{\frac{m}{gc}} \tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right) \quad \checkmark$$

note that t_f does not make the denominator of $v(t) = 0$ so I'm safe. ✓

Thus, the maximum height of the particle is:

$$y_{max} = y(t_f) = \frac{m}{c} \ln\left[\frac{\cos\left(\tan^{-1}\left[V_0\sqrt{\frac{c}{mg}}\right] - \tan^{-1}\left[V_0\sqrt{\frac{c}{mg}}\right]\right)}{\cos\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)}\right]$$

$$= \frac{m}{c} \ln\left[\frac{1}{\cos\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)}\right]$$ ✓



$$y_{max} = \frac{m}{c} \ln\left(\frac{\sqrt{mg}}{\sqrt{mg + cV_0^2}}\right) = \frac{m}{c} \ln\left(\sqrt{\frac{mg}{mg + cV_0^2}}\right)$$ ✓ good job

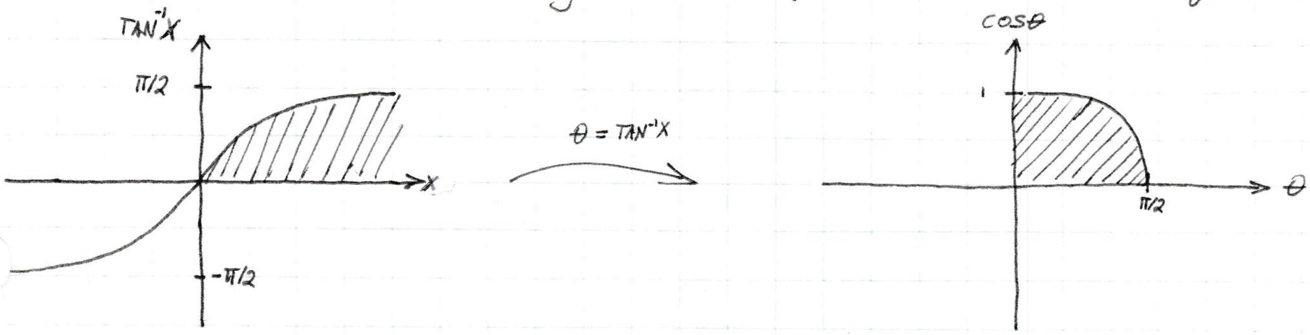
Prelude to part C

I will be expanding $y(t)$, the position function.

$$y(t) = \frac{m}{c} \ln \left| \frac{\cos\left(\tan^{-1}\left[V_0\sqrt{\frac{c}{mg}}\right] - t\sqrt{\frac{gc}{m}}\right)}{\cos\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)} \right|$$

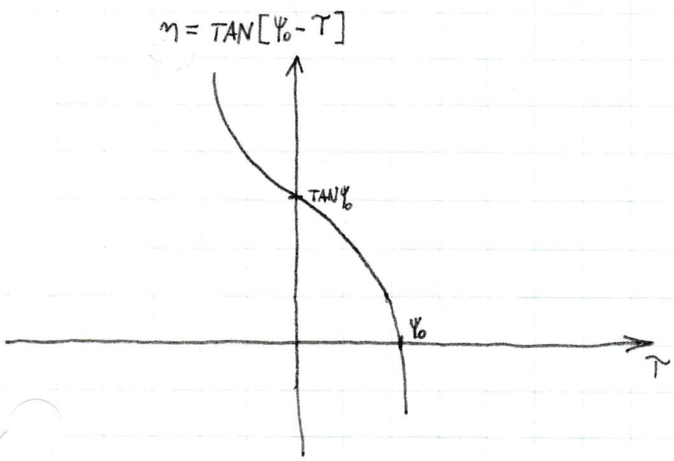
I will argue that the argument of the log is always a positive number and therefore the absolute value signs are not needed.

Consider the denominator: $\cos\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)$. Since $V_0\sqrt{\frac{c}{mg}} = \frac{V_0}{V_T}$ is always a positive number, $\tan^{-1}$ will return an angle between 0 and $\pi/2$ to cos. Of course, $\cos x$ for $0 \le x \le \frac{\pi}{2}$ is non negative. Thus the denominator is positive.



Now I'll argue that the numerator is always positive. The numerator is in the form of $\cos[\Psi_0 - T]$ where $\Psi_0 = \tan^{-1}\left(\frac{V_0}{V_T}\right)$. Since $V_0/V_T > 0$, $0 \le \Psi_0 \le \frac{\pi}{2}$

Now for $T < 0$ we haven't thrown the ball yet and for $T > \Psi_0$ η becomes negative. However, we already noted that this particular solution is valid only when the ball is on its way up, where $\eta > 0$ so T cannot become greater than Ψ_0. Thus, the quantity $\Psi_0 - T$ can be at most $\frac{\pi}{2}$ but must be greater than 0. Thus $\cos(\Psi_0 - T) > 0$ and we can rewrite $y(t)$ as:



good statement.

$$y(t) = \frac{m}{c} \ln \left[\frac{\cos\left(\tan^{-1}\left[V_0\sqrt{\frac{c}{mg}}\right] - t\sqrt{\frac{gc}{m}}\right)}{\cos\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)} \right]$$

part c I will expand the position function.

$$y(t) = \frac{m}{c} \ln\left[\frac{\cos\left(\tan^{-1}\left[V_0\sqrt{\frac{c}{mg}}\right] - t\sqrt{\frac{gc}{m}}\right)}{\cos\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)} \right]$$

Using $\cos(A-B) = \cos A \cos B + \sin A \sin B$

$$y(t) = \frac{m}{c} \ln\left[\frac{\cos\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)\cos\left(t\sqrt{\frac{gc}{m}}\right) + \sin\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)\sin\left(t\sqrt{\frac{gc}{m}}\right)}{\cos\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)} \right]$$

$$= \frac{m}{c} \ln\left[\cos\left(t\sqrt{\frac{gc}{m}}\right) + \tan\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)\sin\left(t\sqrt{\frac{gc}{m}}\right) \right]$$

$$= \frac{m}{c} \ln\left[\cos\left(t\sqrt{\frac{gc}{m}}\right) + V_0\sqrt{\frac{c}{mg}}\sin\left(t\sqrt{\frac{gc}{m}}\right) \right]$$

The goal is to get first order corrections. I will begin to expand the trig stuff. To be on the safe side I'll use up to and including third order terms and throw away what I obviously don't want. Actually, I see that the 3rd order SIN term in conjunction with the $\sqrt{c}$ sitting outside the SIN term will produce a c^3. When I expand the log I'll get a c^2 — so, nix the 3rd order SIN term.

$$= \frac{m}{c} \ln\left[1 - \frac{t^2 gc}{2m} + \frac{1}{24}t^4\left(\frac{gc}{m}\right)^2 + V_0\sqrt{\frac{c}{mg}}\left(t\sqrt{\frac{gc}{m}} - \frac{1}{6}t^3\left[\frac{gc}{m}\right]^{3/2} \right) \right]$$

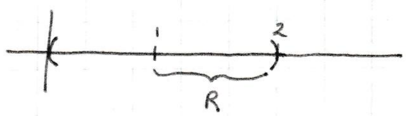
$$= \frac{m}{c} \ln\left[1 + c\left(\frac{V_0 t}{m} - \frac{gt^2}{2m}\right) + c^2\left(\frac{g^2t^4}{24m^2} - \frac{V_0 g t^3}{6m^2}\right) \right]$$

Now the expansion for $\ln(1+x)$ is:

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots + \ldots$$

oh?

Since c is small I won't discuss convergence issues (at least not without getting paid by someone. Necessarily, $\ln(x+1)$ has a radius of convergence of 1 since $\ln 0$ is undefined, and the radius of convergence is centred on the point about which the series is computed. I'll just assume that the $c(\ ) + c^2(\ )$ term is somewhere between 0 and 2.



In the log expansion, $x = c(1) + c^2(2)$

Certainly the 1st order term is needed, but what of the $\frac{1}{2}x^2$ term? The point is to keep C^2 terms since they'll end up being C after multiplication by the outside m/c factor.

Indeed I'll get something that looks like: $\frac{1}{2}\left[c^2(1)^2 + 2c^3(1)(2) + c^4(2)^2\right]$ notice the C^2 term? It's needed, by the critereon outlined in the previous paragraph. so:

$$y(t) = \frac{m}{c}\left[c\left(\frac{v_0 t}{m} - \frac{gt^2}{2m}\right) + c^2\left(\frac{g^2t^4}{24m^2} - \frac{v_0 g t^3}{6m^2}\right) - \frac{1}{2}c^2\left(\frac{v_0 t}{m} - \frac{gt^2}{2m}\right)^2\right]$$

$$= v_0 t - \frac{1}{2}gt^2 + \frac{c}{2m}\left[\frac{g^2t^4}{12} - \frac{v_0 g t^3}{3}\right] - \frac{c}{2m}\left[v_0^2t^2 - v_0 g t^3 + \frac{1}{4}g^2t^4\right]$$

$$= v_0 t - \frac{1}{2}gt^2 + \frac{c}{2m}\left[g^2t^4\left(\frac{1}{12} - \frac{1}{4}\right) + v_0 g t^3\left(1 - \frac{1}{3}\right) - v_0^2t^2\right]$$

$$= \underbrace{v_0 t - \frac{1}{2}gt^2}_{\text{Result sans air resistance}} + \underbrace{c\left[\frac{v_0 g t^3}{3m} - \frac{g^2t^4}{12m} - \frac{v_0^2t^2}{2m}\right]}_{\text{first order correction due to air resistance (includes the } c)}$$

✓ nice job

d) From before we had $V_{down} = \sqrt{\frac{mg}{c}}\, TANH\left(t\sqrt{\frac{gc}{m}}\right)$

$$Y_{down} = \sqrt{\frac{mg}{c}} \int_{t_f}^{t_+} TANH\left(t\sqrt{\frac{gc}{m}}\right) dt$$

A few words. I initially manhandled my non "limit" (indefinate) integral into having the correct expression for $Y_{down}(t_f) = Y_{max}$. I realized afterward that the more elegant way is to use limits. The integration to find Y_{down} (hereafter $Y_d(t)$) starts at t_f, when the particle reaches Y_{max}. It ends at some arbitrary time after t_f which I call t_+. I'm stipulating that $t_+ < t_{impact}$ since if the particle did hit the ground (if $t_+ > t_{impact}$) the differential equation which started this whole thing becomes invalid. ✓

$$Y_d(t) = \frac{m}{c}\int_{t=t_f}^{t=t_+} TANH\, u\, du = \frac{m}{c}\int_{t=t_f}^{t=t_+} \frac{dw}{w}$$

$w = COSH\, u$
$dw = SINH\, u\, du$

$$= \frac{m}{c} \ln\left| COSH\, u\right|_{t=t_f}^{t=t_+} + Y_0$$

$$= \frac{m}{c} \ln\left| COSH\left(t\sqrt{\frac{gc}{m}}\right)\right|_{t=t_f}^{t=t_+} + Y_0 \quad ✓$$

Notes:

1. $COSH\, X > 0 \ \forall X$ so the abs val sign is not needed. ✓
2. Since integration started at $t = t_f$, the initial height, Y_0, is equal to y_{max}.

$$Y_d(t) = \frac{m}{c}\ln\left[COSH\left(t\sqrt{\frac{gc}{m}} - TAN^{-1}\left[V_0\sqrt{\frac{c}{mg}}\right]\right)\right] + \frac{m}{c}\ln\left[\frac{mg}{\sqrt{mg+cV^2}}\right] \quad ✓$$

3. t_+ is just t. I named it t_+ so as not to confuse it with variable of integration (dummy variable in reverse). ✓

Using Y_d I can find the time of impact, t_i (this notation is as bad as it gets. on the other hand I've always preferred t_0 to t_i as an initial time).

$$Y_d \overset{set}{\equiv} 0 \Rightarrow \cosh\left(t_i\sqrt{\frac{gc}{m}} - \tan^{-1}\left[V_0\sqrt{\frac{c}{mg}}\right]\right) = -\sqrt{\frac{mg}{mg+cV^2}}$$

$$t_i\sqrt{\frac{gc}{m}} = -\cosh^{-1}\left(\sqrt{\frac{mg}{mg+cV^2}}\right) + \tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right)$$

$$t_i = \sqrt{\frac{m}{gc}}\left[\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right) - \cosh^{-1}\left(\sqrt{\frac{mg}{mg+cV^2}}\right)\right]$$

Now I'll use the above expression for t_i to find the speed at impact, V_i.

$$V_{down}(t_i) = V_i = \sqrt{\frac{mg}{c}}\tanh\left[\sqrt{\frac{gc}{m}}\sqrt{\frac{m}{gc}}\left(\tan^{-1}\left[V_0\sqrt{\frac{c}{mg}}\right] - \cosh^{-1}\left(\sqrt{\frac{mg}{mg+cV^2}}\right)\right)\right]$$

$$V_i = \sqrt{\frac{mg}{c}}\tanh\left[\tan^{-1}\left(V_0\sqrt{\frac{c}{mg}}\right) - \cosh^{-1}\left(\sqrt{\frac{mg}{mg+cV^2}}\right)\right]$$