Physics 330 Home work Problem # 11.

got the loose sheets!

A thin rigid hoop of mass m and radius h rolls without slipping around an horizontal circle of radius R on the inside of a vertical, right circular cylindrical wall. The speed of the hoop's cm is V, presumed great enough to maintain force balance. Including angular momentum due to rotation of the hoop within its plane, but neglecting any angular momentum of the hoop due to rotation of its plane, derive an for the angle of the hoop's plane with the horizontal. Solve the equation for the angle keeping terms of order h^2/R^2, and identify the terms that express the effect of the hoop's internal angular momentum on the result. Discuss qualitatively the effect of including effects of rotation of the hoop's plane on the calculation of the angle.

specifically what diff?

$$\eta = \frac{V}{\sqrt{gR}}$$

$$0 = \frac{1}{2}\eta^2 - (1-\cos\theta) + \mu_k \sin\theta - \mu_k \int_0^{\theta} \eta^2 \, d\theta$$

Is L ∥ to $\bar{\omega}$

why is L not ∥ to $\bar{\omega}$

what direction?

θ inc or dec?

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Physics 330
Analytical Mechanics
Problem # 11

+ −

very nice, I've got to ding you a bit for forgetting the last part of the question.

last 2 questions.
compare θ_2 with another good student.
pessimist!

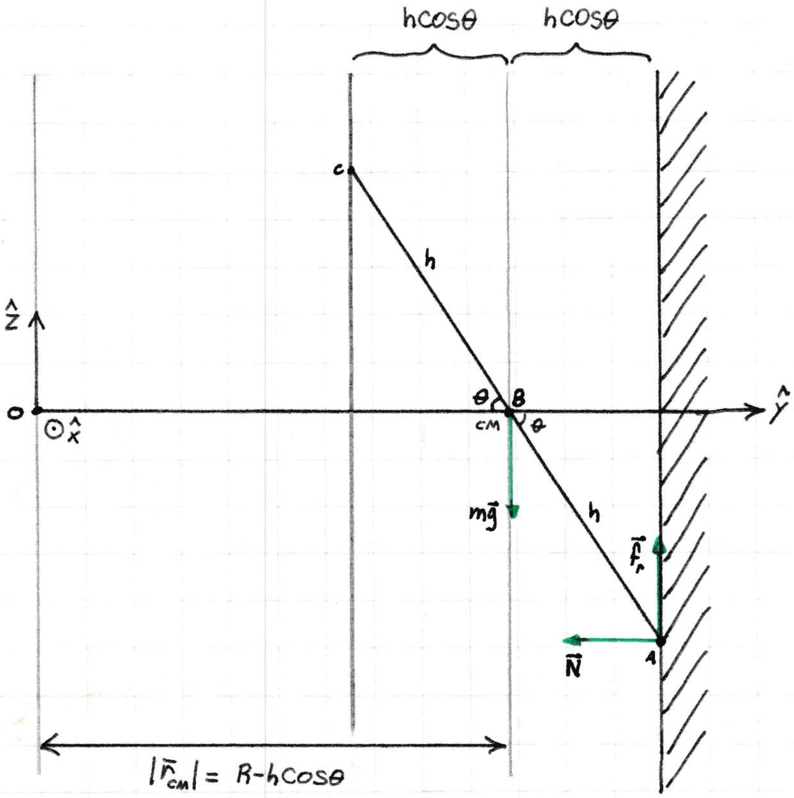
The Hoop

A thin rigid hoop of mass m and radius h rolls without slipping around an horizontal circle of radius R on the inside of a vertical, right circular cylindrical wall. The speed of the hoop's CM is V_{cm}, presumed great enough to maintain force balance.

Including angular momentum due to the rotation of the hoop within its plane, but neglecting any angular momentum of the hoop due to rotation of its plane, derive an equation for the angle of the hoop's plane with the horizontal. Solve the equation for the angle keeping terms of order h^2/R^2, and identify the terms that express the effect of the hoop's internal angular momentum on the result.

Discuss qualitatively the effect of including effects of rotation of the hoop's plane on the calculation of the angle.

Problem #11



?

In cylindrical spherical coordinates,

$\vec{r}_{cm} = (R - h\cos\theta)\hat{\rho}$ position vector of hoop's centre of mass

$\vec{p}_{cm} = mV_{cm}\hat{\phi}$

The external angular momentum is the angular momentum of the system as if it were a particle at the location of the CM travelling at $\vec{V}_{cm}$. For this reason I think that "particle term of angular momentum" is a better description (you should review the text!)

$$\vec{L}_{o,particle} = \vec{r}_{cm} \otimes \vec{p}_{cm} = (R - h\cos\theta)\hat{\rho} \otimes mV_{cm}\hat{\phi}$$

$$= mV_{cm}(R - h\cos\theta)[\hat{\rho} \otimes \hat{\phi}]$$

$$= mV_{cm}(R - h\cos\theta)\hat{z}$$ ✓

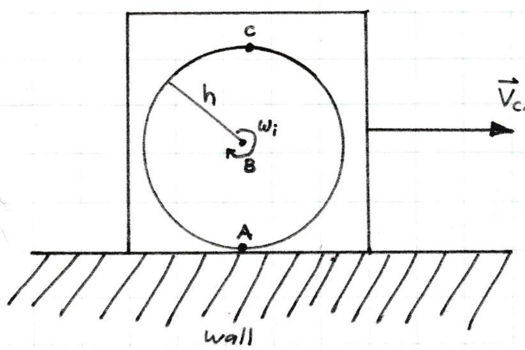
Notice that $\vec{L}_{o,particle}$ is not dependant on time, hence $\frac{d\vec{L}_p}{dt} = 0$. ✓ Therefore, external torque must be zero. $\vec{T}_{ext}$ is computed as if all the forces acting on the hoop occur at the CM. Vertical force balance implies: $\sum F_z = f_r - mg = 0 \implies f_r = mg$

The only net external force is the normal force.

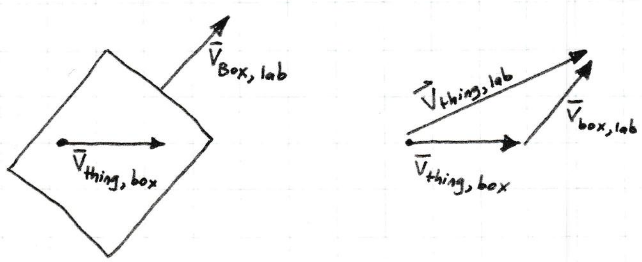
$$\bar{T}_{ext} = \bar{r}_{CM} \otimes \bar{F}_{ext} = (R - h\cos\theta)\hat{\rho} \otimes N(-\hat{\rho})$$
$$= N(R - h\cos\theta)(\hat{\rho} \otimes \hat{\rho})$$
$$= 0$$

as expected.

Internal Term



Now I'll consider the internal angular momentum. Picture the hoop in its CM frame (recall that the box to the left which represents the frame is at an angle of $\frac{\pi}{2} - \theta$ to the wall). The box travels at a velocity $\vec{V}_{cm}$ since the hoop translates at a velocity $\vec{V}_{cm}$ in the laboratory frame. Now recall the concept of relative velocity:



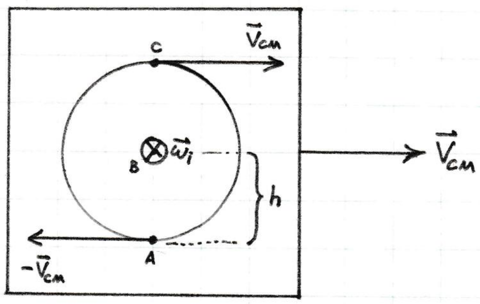
$$\vec{V}_{thing,\,lab} = \vec{V}_{thing,\,box} + \vec{V}_{box,\,lab} \qquad (i)$$

The next argument will rely on this fact. The goal here is to find the internal angular speed, ω_i. ω_i can be used to find the internal angular momentum, $\vec{L}_{int}$

The centre of the hoop travels at a velocity of $\vec{V}_{cm}$. By eq (i), since the box already travels at $\vec{V}_{cm}$, point B must be at rest in the lab frame.

or A?

Now point A will be instantaneously at rest wrt the wall in the laboratory frame. This is equivalent to saying that the hoop is rolling w/o slipping. However, it's travelling at V_{cm} along with everything else in the CM frame. To make it at rest in the lab frame wrt the wall it must be travelling at $-\vec{V}_{cm}$ in the CM frame.



Since we know that that the hoop is rotating about its CM in its CM frame (rotating about point B), point C must be moving at a velocity $\vec{V}_{cm}$.

In the CM frame, the speed of point A is:

$$V_A = \omega_i h = V_{cm} \qquad \text{(ii)}$$

Now let's flip back to the point of view in the laboratory frame. We know that point B is a distance $R - h\cos\theta$ from the origin and travels at a speed V_{cm}. Thus:

$$V_B = V_{cm} = \omega_o (R - h\cos\theta) \qquad \text{(iii)}$$

Equating the 2 expressions for V_{cm} (eq's ii + iii)

$$V_{cm} = V_{cm} \Rightarrow \omega_i h = \omega_o (R - h\cos\theta)$$

$$\boxed{\omega_i = \frac{R - h\cos\theta}{h}\,\omega_o = \frac{V_{cm}}{h}} \qquad \text{(iv)}$$

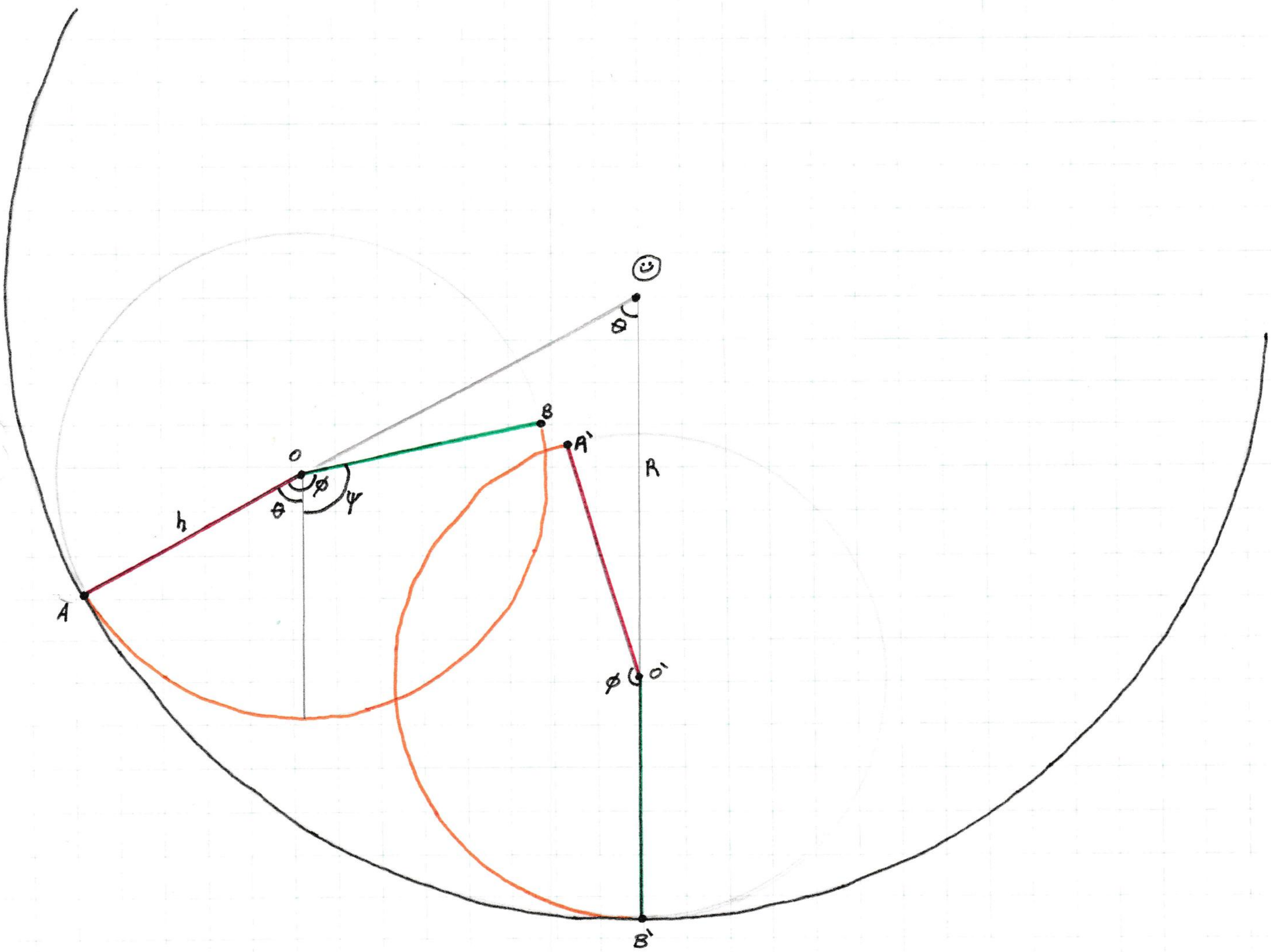
Now I'll attempt to find ω_i in a more complicated manner since I have nothing better to do on a saturday night.

right.

The picture below shows the hoop with its plane $\perp$ to the cylinder at some time t and then again to the right at time $t + \Delta t$.

Clearly the hoop is rotating about its centre, O, in a CW manner. I was very careful in drawing this diagram, using a compass, ruler and dental floss (to mete out arc lengths). Points and lines which are the same (but changed position in time) have been colour coded and have the same name, with prime indicating a later time. eg - point A\` represents point A a time Δt later. Thus, the 2 orange arc lengths connecting A with

really got your teeth into it.



A\` and B with B\` are of equal length and represent that part of the hoop which was travelled on.

The centre of the hoop completed an angle θ wrt point ☺ during the time interval Δt. (I couldn't bear to call it O\`\`\`).

☺ to you too

The arc length AB is equal to $R\theta$ – this follows from simple geometry, $s = r\theta$. By the same token this distance is also $h\phi$.

$$R\theta = h\phi \qquad (v)$$

Now if I can relate θ to ω_o and ϕ to ω_i, I'll have a relationship between ω_o and ω_i. This relationship should be equivalent to (iv).

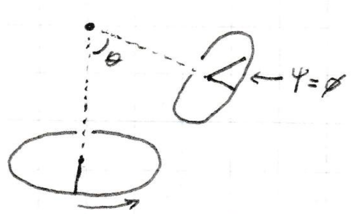
The external angle θ is simple enough: $\theta = \omega_o \Delta t$. The internal angle ϕ looks straight forward also, at 1st glance. (and 2nd glance, 3rd glance, ..., N-1 glance). On the N'th glance (the glance with you standing over my shoulder), ϕ is not simply $\omega_i \Delta t$. The problem is that ϕ is not just due to the hoop's rotation – it's a combination of the angle by which the hoop would've rotated if it's CM were stationary wrt ☺ plus the angle through which O moves, angle θ; from the diagram $\phi = \psi + \theta$. θ is the angle through which the CM moves and has nothing to do with the hoop's external angular speed. ψ is the angle through which the hoop rotates wrt its CM. These last 2 sentances provide the key to understanding what's going on.

$$R\theta = h(\psi + \theta)$$

$$R\omega_o \Delta t = h(\omega_i \Delta t + \omega_o \Delta t) \qquad (vi)$$

$$\Rightarrow \quad \omega_i = \frac{\omega_o(R-h)}{h} \qquad (vii)$$

Of course, (vii) is not equivalent to (iv). There is one last complication involved. The setup considered had the hoop rotating in the same plane that the CM moves in. Had the hoop been $\perp$ to the plane of the moving CM, the angle by which the CM moves would not have contributed to the angle by which the hoop rotated wrt the wall. The 2 rotations would result in different types of motion – independant motion. θ would not contribute to the total angle by which the hoop rotates wrt the wall.



Therefore, the contribution of $h\omega_o \Delta t$ to $R\theta$ must be modified so that it is equal to itself when the hoop is $\perp$ to the wall and 0 when the hoop is

|| to the wall. Whoops - I really shot myself in the foot by using θ to be 2 different angles. I hope all of this is clear.

Yup - It come as a bit of a surprise how popular this approach was, given its complexity.

Considering the "boundary conditions" of this effect (ie: full effect when hoop is $\perp$ to wall, zero effect when ||) $\cos\theta$ does the right thing (see 1st pg for how I originally defined θ. I will stick to it from now on).

Now that's a pretty hand waving argument, and I was going to leave it at that.

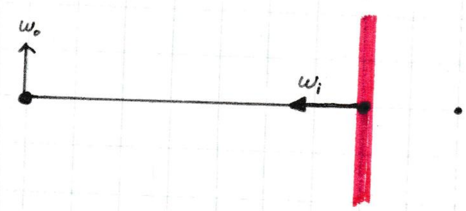
But I think that I figured it out for real, almost.

Diagram A:

Full effect

Diagram A shows the hoop $\perp$ to the wall. As mentioned, the effect is maximized here. Diagram B shows the hoop || to the wall -

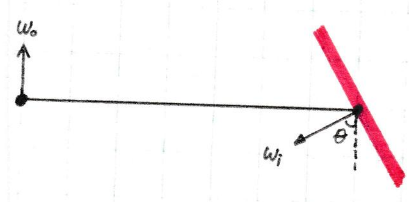
Diagram B:



No effect

Something happens when $\bar{w}_o$ has a component in the direction anti-parallel to $\bar{w}_i$ - this component leads to a ϕ which is larger than $w_i t$.

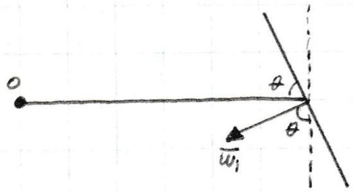
When expanding
$R\theta = h\phi \Rightarrow Rw_o = h(w_i + w_o)$
we want to consider the component of $\bar{w}_o$ which is antiparallel to $\bar{w}_i$. thus:

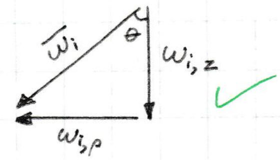


$$R\theta = h\phi \quad \Rightarrow \quad Rw_o = h(w_i + w_o\cos\theta)$$

It's a shame that school's over. I would've liked to have given this more thought. It's looking good.

Well, in any event, $|\vec{\omega}_i|$ is now a known quantity. Once I find its direction, L_{int} will immediately follow.





From the diagram, $\vec{\omega}_i = -\omega_i(\cos\theta\,\hat{z} + \sin\theta\,\hat{\rho})$

I claim that the hoop is rotationally symmetric wrt its axis of rotation. Therefore L_{int} is // to $\vec{\omega}_{int}$ and

$$\vec{L}_{int} = I_{hoop}\,\vec{\omega}_{int}$$

is valid. See Physics, the Nature of things, figure 13.11. Hence:

My god we made it into the literature!

$$\vec{L}_{int} = I_{hoop}\,\vec{\omega}_{int} = -mh^2\left[\frac{R-h\cos\theta}{h}\right]\omega_0(\cos\theta\,\hat{z} + \sin\theta\,\hat{\rho})$$

$$= -mh(R-h\cos\theta)\omega_0(\cos\theta\,\hat{z} + \sin\theta\,\hat{\rho})$$

Now I'll calculate $\frac{d\vec{L}_{int}}{dt}$, and then $\vec{\tau}_{int}$. Note $\theta \neq \theta(t)$

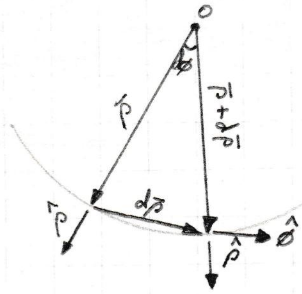
$$\frac{d\vec{L}_{int}}{dt} = -mh(R-h\cos\theta)\omega_0\frac{d}{dt}[\cos\theta\,\hat{z} + \sin\theta\,\hat{\rho}]$$

Now $\hat{z}$ does not change with time - it's a fixed unit vector. That's one of the beautiful things about the Cartesian coordinate system. Thus, $\frac{d}{dt}[\cos\theta\,\hat{z}] = 0$

Just you wait Enry Iggins!

$$\frac{d\vec{L}_{int}}{dt} = -mh(R-h\cos\theta)\omega_0\sin\theta\,\frac{d\hat{\rho}}{dt}$$

What is $\frac{d\hat{\rho}}{dt}$?



From the diagram and $d\bar{s} = R\,d\theta$

$$d\bar{\rho} = \rho\, d\phi\, \hat{\phi} \quad \Rightarrow \quad d\hat{\rho} = d\phi\, \hat{\phi}$$

$$\Rightarrow \quad \frac{d\hat{\rho}}{dt} = \frac{d\phi}{dt}\hat{\phi}$$

Now $\frac{d\phi}{dt}$ is an angular speed, but is it $\bar{\omega}_i$, $\bar{\omega}_o$ or a combination of both? The coordinate $\hat{\rho}$ describes a position wrt O, the centre of the cylinder. Thus, ω must be the angular speed about O, which is ω_o. Thus:

$$\frac{d\hat{\rho}}{dt} = \omega_o\, \hat{\phi}$$

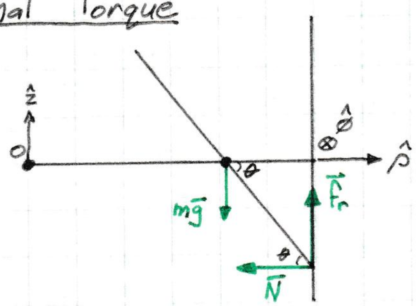
and

$$\frac{d\vec{L}_{int}}{dt} = -mh(R-h\cos\theta)\,\omega_o \sin\theta\, \frac{d\hat{\rho}}{dt}$$

$$= -mh(R-h\cos\theta)\,\omega_o^2 \sin\theta\, \hat{\phi}$$

$$= -\frac{mh V_{cm}^2 \sin\theta\, \hat{\phi}}{R-h\cos\theta} \quad \checkmark$$

<u>Internal Torque</u>



Internal torque is computed wrt the hoop's CM, so the hoop's weight (which acts at the CM) does not produce an internal torque.

$$\vec{T}_{int} = h\sin\theta\, N \phi - h\sin\left(\frac{\pi}{2}-\theta\right) f_r\, \hat{\phi}$$

① As mentioned previously, $\sum F_z$ gave us $f_r = mg$.

② $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

③ The normal force is the force that accelerates the hoop's CM centripetally about O. Therefore,

$$\sum F_\rho = -N = -\frac{mV_{cm}^2}{R-h\cos\theta} \quad \Rightarrow \quad N = \frac{mV_{cm}^2}{R-h\cos\theta} \quad \checkmark$$

so: $$\vec{T}_{int} = h\sin\theta\left(\frac{m V_{cm}^2}{R-h\cos\theta}\right)\hat{\phi} - h\cos\theta\, mg\,\hat{\phi}$$

$$= \frac{mhV_{cm}^2\sin\theta}{R-h\cos\theta}\hat{\phi} - mgh\cos\theta\,\hat{\phi}$$

And here's the climax of the problem: $\vec{T}_{int} \overset{set}{\underset{(is)}{\equiv}} \dot{\vec{L}}_{int}$

$$-\frac{mhV_{cm}^2\sin\theta}{R-h\cos\theta}\hat{\phi} = \frac{mhV_{cm}^2\sin\theta}{R-h\cos\theta}\hat{\phi} - mgh\cos\theta\,\hat{\phi}$$

or, rearranging terms,

$$\tan\theta = \frac{g(R-h\cos\theta)}{2V_{cm}^2} \qquad \text{(viii)}$$

✓ well done

Of zeroth order

$$\tan\theta = \frac{gR}{2V^2} \implies \theta_0 = \tan^{-1}\left(\frac{gR}{2V^2}\right)$$

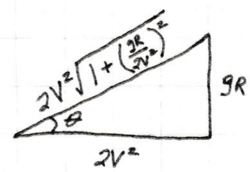
Of first order

Note that $\tan\theta_0$ appears on both sides of (vi), so we can simplify (vi) by subtracting it out.

$$\sec^2\theta_0(\theta-\theta_0) + \sec^2\theta_0\tan\theta_0(\theta-\theta_0)^2 = -\left(\frac{gR}{2V^2}\right)\left(\frac{h}{R}\right)\cos\theta_0 + \left(\frac{gR}{2V^2}\right)\left(\frac{h}{R}\right)\sin\theta_0(\theta-\theta_0) \quad \text{(vii)}$$

Now (vii) is our working equation. The equation of order ~~a~~ one is:

$$\sec^2\theta_0(\theta-\theta_0) = -\left(\frac{gR}{2V^2}\right)\left(\frac{h}{R}\right)\cos\theta_0$$



Let $\otimes \equiv 1 + \left(\frac{gR}{2V^2}\right)^2$. Then:

$$\otimes\left(\frac{h}{R}\theta_1\right) = -\left(\frac{gR}{2V^2}\right)\left(\frac{h}{R}\right)\otimes^{-1/2}$$

$$\theta_1 = -\left(\frac{gR}{2V^2}\right)\otimes^{-3/2}$$

$$= -\left(\frac{gR}{2V^2}\right)\left[1 + \left(\frac{gR}{2V^2}\right)^2\right]^{-3/2}$$

Of order two

Now we move up to 2nd order in (ii). Going back to (vii) and keeping vigilance against terms of order three or greater,

$$\otimes\left[\frac{h}{R}\theta_1 + \left(\frac{h}{R}\right)^2\theta_2\right] + \otimes\left(\frac{gR}{2V^2}\right)\left[\frac{h}{R}\theta_1 + \left(\frac{h}{R}\right)^2\theta_2\right]^2 = -\frac{gR}{2V^2}\left(\frac{h}{R}\right)\otimes^{-1/2} + \left(\frac{gR}{2V^2}\right)^2\left(\frac{h}{R}\right)\otimes^{-1/2}[\theta-\theta_0]$$

nothing else survives

$$\cancel{-\left(\frac{gR}{2V^2}\right)\left(\frac{h}{R}\right)\otimes^{-1/2}} + \otimes\left(\frac{h}{R}\right)^2\theta_2 + \otimes\left(\frac{gR}{2V^2}\right)\left(\frac{h}{R}\theta_1\right)^2 = \cancel{-\frac{gR}{2V^2}\left(\frac{h}{R}\right)\otimes^{-1/2}} + \left(\frac{gR}{2V^2}\right)^2\left(\frac{h}{R}\right)\otimes^{-1/2}[\theta-\theta_0]$$

$$\otimes\left(\frac{h}{R}\right)^2\theta_2 + \otimes\left(\frac{gR}{2V^2}\right)\left(\frac{h}{R}\right)^2\left(\frac{gR}{2V^2}\right)^2\otimes^{-3} = \left(\frac{gR}{2V^2}\right)^2\left(\frac{h}{R}\right)\otimes^{-1/2}\left[\frac{h}{R}\theta_1 + \cancelto{0}{\left(\frac{h}{R}\right)^2\theta_2}\right]$$

$$\otimes\left(\frac{h}{R}\right)^2\theta_2 = -\otimes^{-2}\left(\frac{gR}{2V^2}\right)^3\left(\frac{h}{R}\right)^2 + \left(\frac{gR}{2V^2}\right)^2\left(\frac{h}{R}\right)^2\otimes^{-1/2}\left(-\frac{gR}{2V^2}\otimes^{-3/2}\right)$$

$$\otimes\theta_2 = -\left(\frac{gR}{2V^2}\right)^3\cancel{\left(\frac{h}{R}\right)^2}\otimes^{-2} - \left(\frac{gR}{2V^2}\right)^3\otimes^{-2}$$

$$\theta_2 = -2\left(\frac{gR}{2V^2}\right)^3\otimes^{-2}$$

$$= -\frac{1}{4}\left(\frac{gR}{V^2}\right)^3\left[1+\left(\frac{gR}{2V^2}\right)^2\right]^{-2}$$

so, the .... uhhh... oh yeah! The hoop is at an angle

$$\theta \approx \theta_0 + \frac{h}{R}\theta_1 + \left(\frac{h}{R}\right)^2\theta_2$$

$$= \mathrm{TAN}^{-1}\left(\frac{gR}{2V^2}\right) - \left(\frac{gR}{2V^2}\right)\left(\frac{h}{R}\right)\left[1+\left(\frac{gR}{2V^2}\right)^2\right]^{-3/2} - \frac{1}{4}\left(\frac{gR}{V^2}\right)^3\left(\frac{h}{R}\right)^2\left[1+\left(\frac{gR}{2V^2}\right)^2\right]^{-2}$$

to the horizontal. See page 1 for definition of θ.

Remember the hoop? ✓ yup. nicely done

(remember the question on qualitative effect of rotation of the plane of the hoop?)