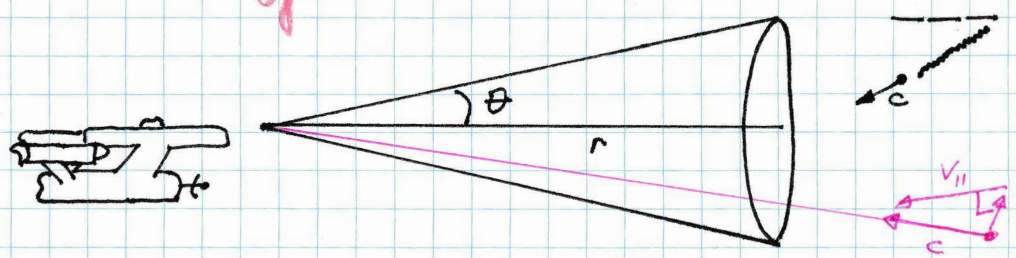
Salzman 3-1

+ good job

Problem 3



Suppose Voyager is at rest and Janeway sets the view screens open to a subtend a cone of half angle θ. The solid angle viewed by the crew is:

$$\Omega = \int d\Omega = \int \frac{dA_\perp}{r^2} = \int_0^{2\pi}\int_0^{\theta} \sin\theta' \, d\theta' \, d\phi$$

$$= 2\pi\left[1-\cos\theta\right]$$

Then, if there are N_0 visible stars the star density will be given by

$$\frac{dN}{d\Omega} \rho = \frac{N_0}{4\pi}$$

better notation since ρ evokes the number per unit volume while you want number per unit solid angle.

And the number of stars viewed within the cone of half angle θ is:

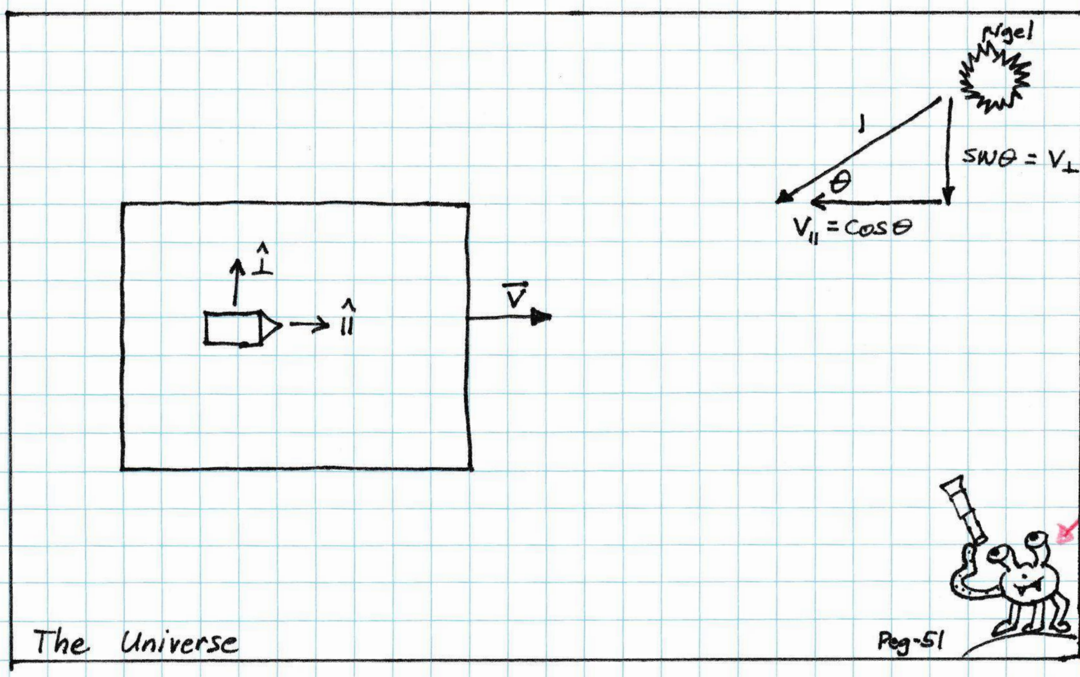
$$N = \frac{dN}{d\Omega}\rho\,\Omega = \frac{N_0}{4\pi}\cdot 2\pi\left[1-\cos\theta\right] = \tfrac{1}{2}N_0\left[1-\cos\theta\right]$$

Now suppose that the ship moves at some speed V. Any photon whose trajectory keeps the photon within the cone will be viewed.

Any photon racing towards the rocket (in the rocket's rest frame) will have a velocity which can be written in terms of $V_\parallel$ and $V_\perp$ where $\parallel$ and $\perp$ denote parallel and perpindicular to the rocket's line of sight. It is the ratio of V_y to V_x that determines the angle of the photon's trajectory with the rocket's line of sight.

However, if we are to consider the photon's trajectory in the rocket frame we need to transform $V_\parallel$ and $V_\perp$ to the rocket frame and we know they don't transform equally.

What I'm about to do is analyze the situation in the plane of the paper. However, I'll keep my discussion coordinate free by talking in terms of $V_\perp$ and $V_\parallel$ to the rocket's forward direction. Rotation of this page about the rocket's velocity vector will generalize this argument to any azimuthal angle desired. well stated

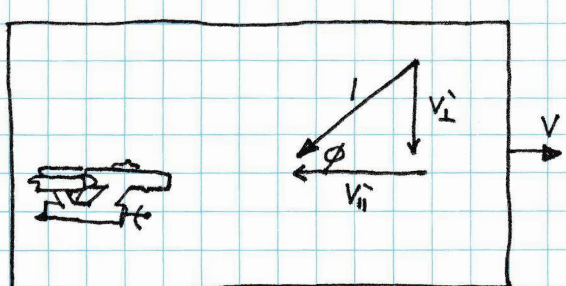


what is your argument for two eyes on the alien?

To an observer on Peg-51 (a very long distance away) the photon has velocity components of:

$$V_\perp = -\sin\theta \qquad V_\parallel = -\cos\theta$$

Now we'll transform into the rocket frame, using the inverse velocity combination formulae.



$$V_\parallel' = \frac{V_\parallel - V}{1 - VV_\parallel} = -\left[\frac{\cos\theta + V}{1 + V\cos\theta}\right]$$

$$V_\perp' = \frac{V_\perp}{\gamma[1 - VV_\parallel]} = -\frac{\sin\theta}{\gamma(1 + V\cos\theta)}$$

The important question is: what does the astronaut see? The important angle is not θ: the astronaut does not measure θ. The astronaut measures ϕ. We can calculate ϕ easily from the diagram.

$$\phi = \tan^{-1}\left[\frac{V_\perp'}{V_\parallel'}\right] = \tan^{-1}\left[\frac{\sin\theta}{\gamma(V+\cos\theta)}\right]$$

Richard said it couldn't be done, but I beg to differ.

$$\tan\phi = \frac{\sin\theta}{\gamma[V+\cos\theta]}$$

Those d - - - d parabola problems finally come in handy.

$$\gamma^2\tan^2\phi\left[V^2+2V\cos\theta+\cos^2\theta\right] = 1-\cos^2\theta$$

$$\cos^2\theta\left[\gamma^2\tan^2\phi+1\right] + 2V\gamma^2\tan^2\phi\cos\theta + \beta^2\gamma^2\tan^2\phi - 1 = 0$$

Solving the quadratic in $\cos\theta$,

$$\cos\theta = \frac{-2V\gamma^2\tan^2\phi \pm \sqrt{4V^2\gamma^4\tan^4\phi - 4[\gamma^2\tan^2\phi+1][V^2\gamma^2\tan^2\phi-1]}}{2(\gamma^2\tan^2\phi+1)}$$

$$= \frac{-V\gamma^2\tan^2\phi \pm \sqrt{\gamma^2\tan^2\phi(1-V^2)+1}}{\gamma^2\tan^2\phi+1}$$

Which root?? Well, as $V\to 0$ we had better recover the Newtonian result $\cos\theta = \cos\phi$! In this limit, $\gamma\to 1$.

$$\lim_{V\to 0}\cos\theta = \frac{0\pm\sqrt{\tan^2\phi(1-0)+1}}{\tan^2\phi+1}$$

$$= \frac{\pm\sqrt{\tan^2\phi+1}}{\tan^2\phi+1}$$

nice work

$$= \pm\cos\phi$$

AHA!!

The positive root!

The astronaut, initially at rest, ends up travelling at some speed V. For a fixed viewing angle $\emptyset$, the fractional increase in stars observed will be:

$$f = \frac{\text{what he sees} - \text{what he saw}}{\text{what he sees}}$$

people differ, often dogmatically over whether sees or saw is appropriate here. I incline toward saw.

$$= \frac{\frac{1}{2}N_0\left\{1-\left[\frac{-V\gamma^2 TAN^2\emptyset+\sqrt{\gamma^2 TAN^2\emptyset(1-V^2)+1}}{\gamma^2 TAN^2\emptyset+1}\right]\right\}-\frac{1}{2}N_0(1-\cos\emptyset)}{\frac{1}{2}N_0\left[1-\left\{\frac{-V\gamma^2 TAN^2\emptyset+\sqrt{\gamma^2 TAN^2\emptyset(1-V^2)+1}}{\gamma^2 TAN^2\emptyset+1}\right\}\right]}$$

Oops – I had failed to notice that inside the square root, we have a nice cancellation of γ^2 and $(1-V^2)$.

Then it is perhaps more impressive when $f \to$ big. Still $f \to 1$ will do.

$$f = \frac{1-\left[\frac{-V\gamma^2 TAN^2\emptyset + SEC\emptyset}{\gamma^2 TAN^2\emptyset+1}\right]+\cos\emptyset-1}{1-\left\{\frac{-V\gamma^2 TAN^2\emptyset + SEC\emptyset}{\gamma^2 TAN^2\emptyset+1}\right\}}$$

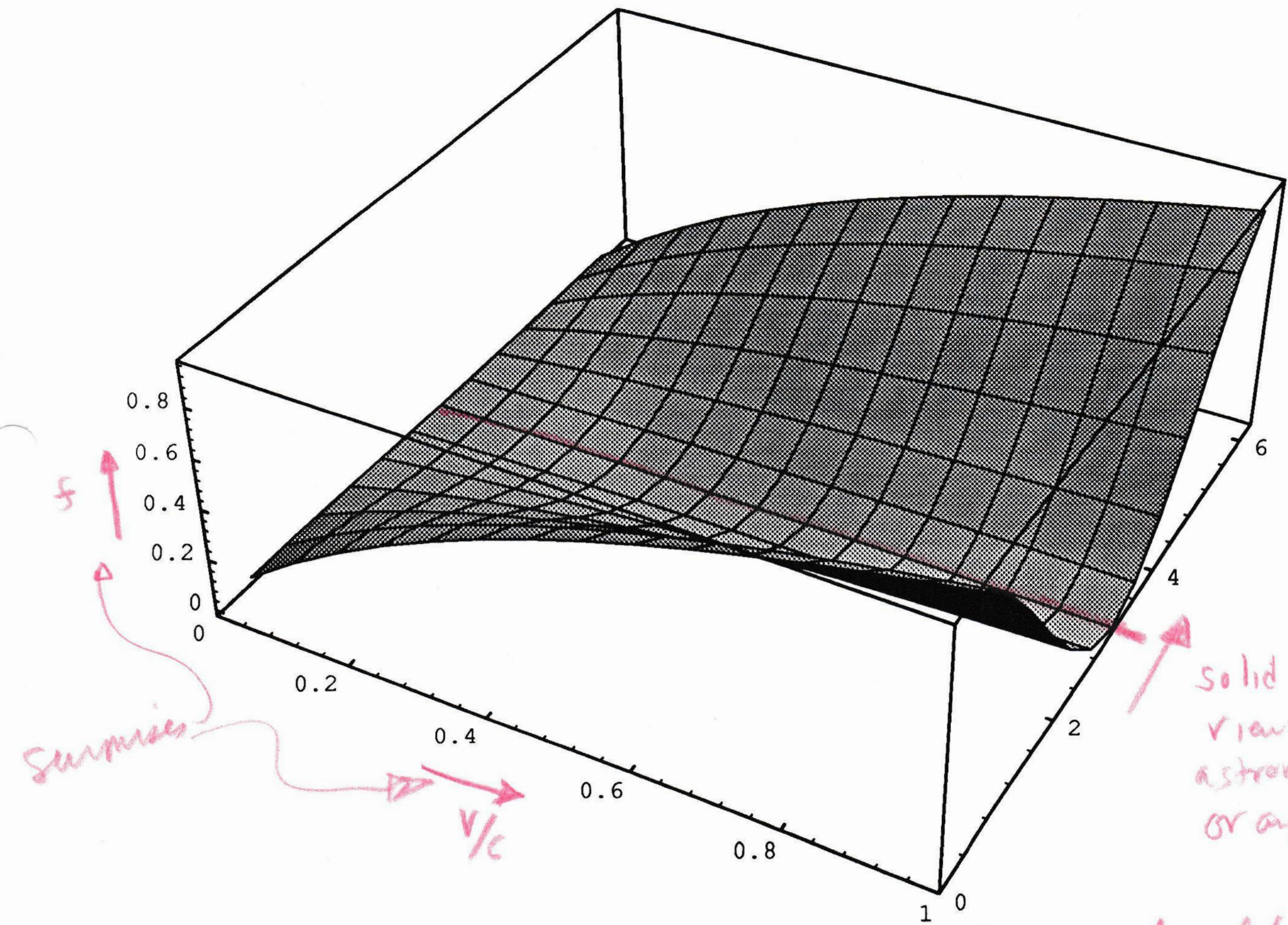
$$= \frac{\frac{V\gamma^2 TAN^2\emptyset - SEC\emptyset}{\gamma^2 TAN^2\emptyset+1}+\cos\emptyset}{\frac{\gamma^2 TAN^2\emptyset+1+V\gamma^2 TAN^2\emptyset - SEC\emptyset}{\gamma^2 TAN^2\emptyset+1}}$$

$$= \frac{V\gamma^2 TAN^2\emptyset - SEC\emptyset + \cos\emptyset(\gamma^2 TAN^2\emptyset+1)}{\gamma^2 TAN^2\emptyset+1+V\gamma^2 TAN^2\emptyset - SEC\emptyset}$$

$$= \frac{V\gamma^2 TAN^2\emptyset - SEC\emptyset + \cos\emptyset(\gamma^2 TAN^2\emptyset+1)}{\gamma^2 TAN^2\emptyset(1+V) - SEC\emptyset + 1}$$

Recap: The next few graphs will represent the fractional increase in the number of stars observed by an astronaut travelling at some speed V over the same astronaut at rest. As V increases, so will the number of observed stars. The graphs show $f(V)$ vs V for different fixed viewing angles $\emptyset$.

Plot3D [f[v,p], {v,0,1}, {p,0,2Pi}]   AspectRatio -> Automatic



solid angle view by astronauts, or angle ϕ?

I'm a bit puzzled about the shape of the plot.

$\lim_{v \to c}$ (what he sees) = N_0 for all solid angles of view,

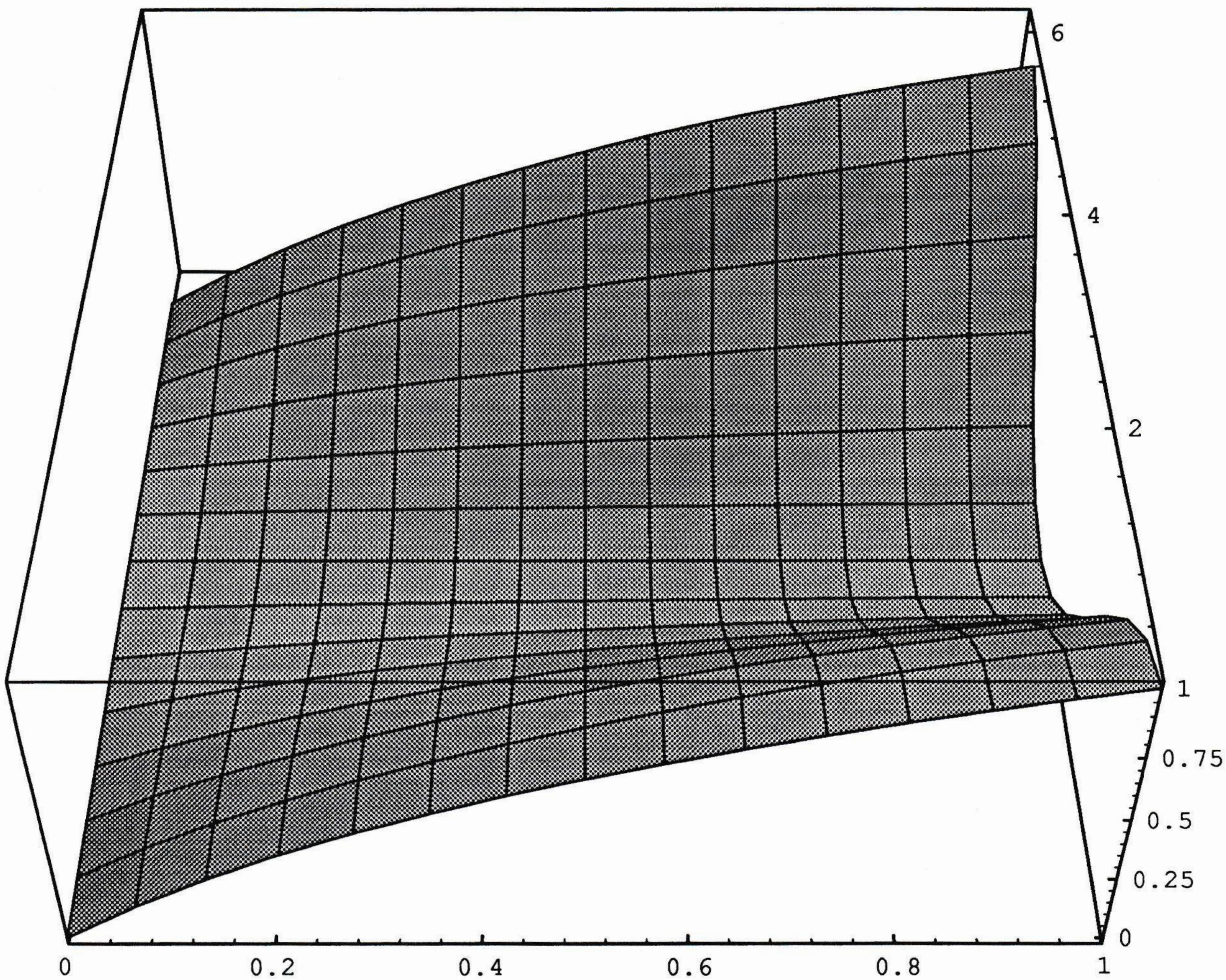
so $\lim_{v \to c} f(p) = \dfrac{N_0 - \frac{N_0}{2}(1-\cos\phi)}{N_0} = \frac{1}{2}(1+\cos\phi)$

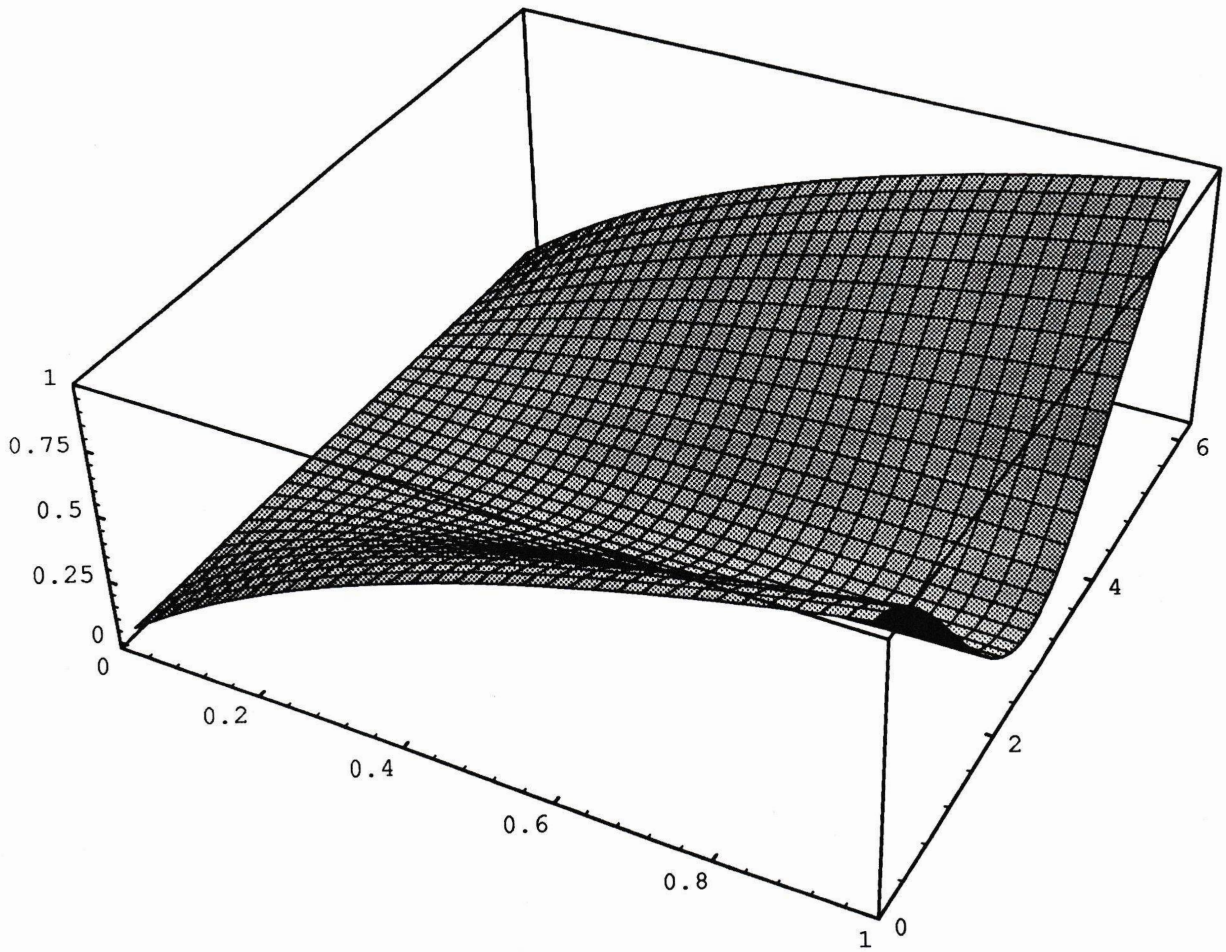
$\to 0$ as $\phi \to \pi$

since astronaut starts out viewing all stars.

If so, the graph should be terminated at the red line.

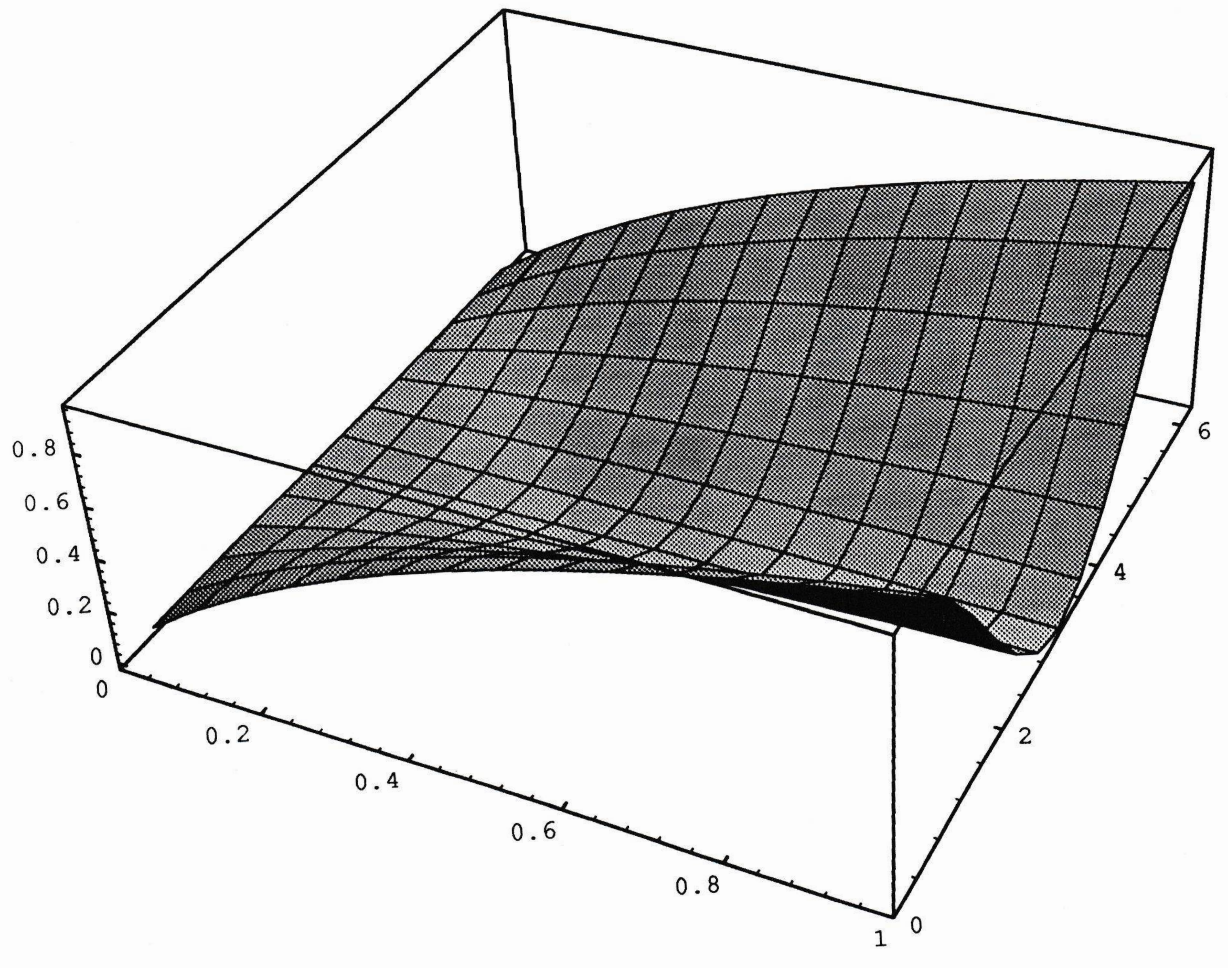
Ah — it looks like you're plotting $\phi > \pi$ which is unphysical.

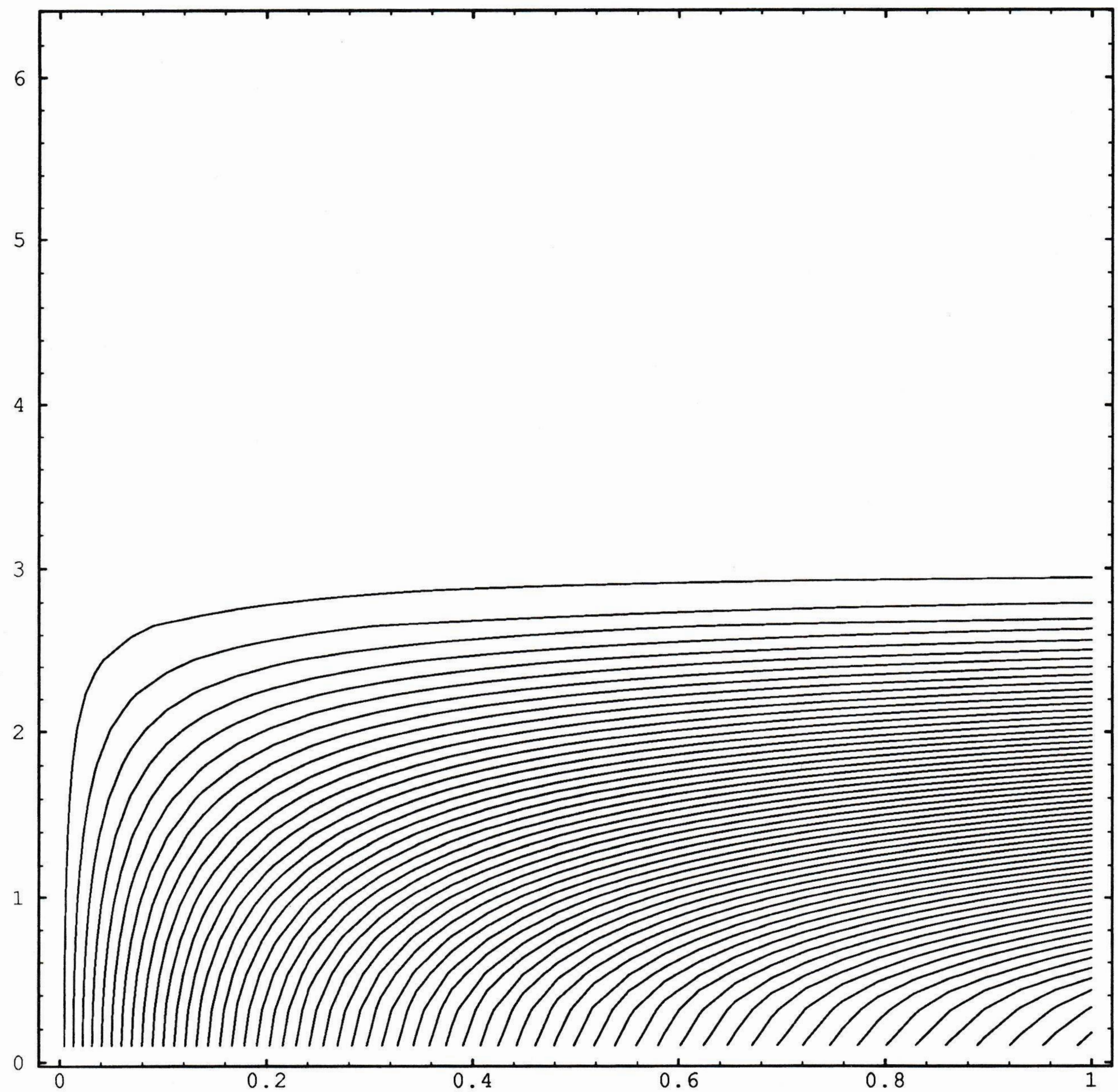

6
4
2
1
0.75
0.5
0.25
0
0
0.2
0.4
0.6
0.8
1



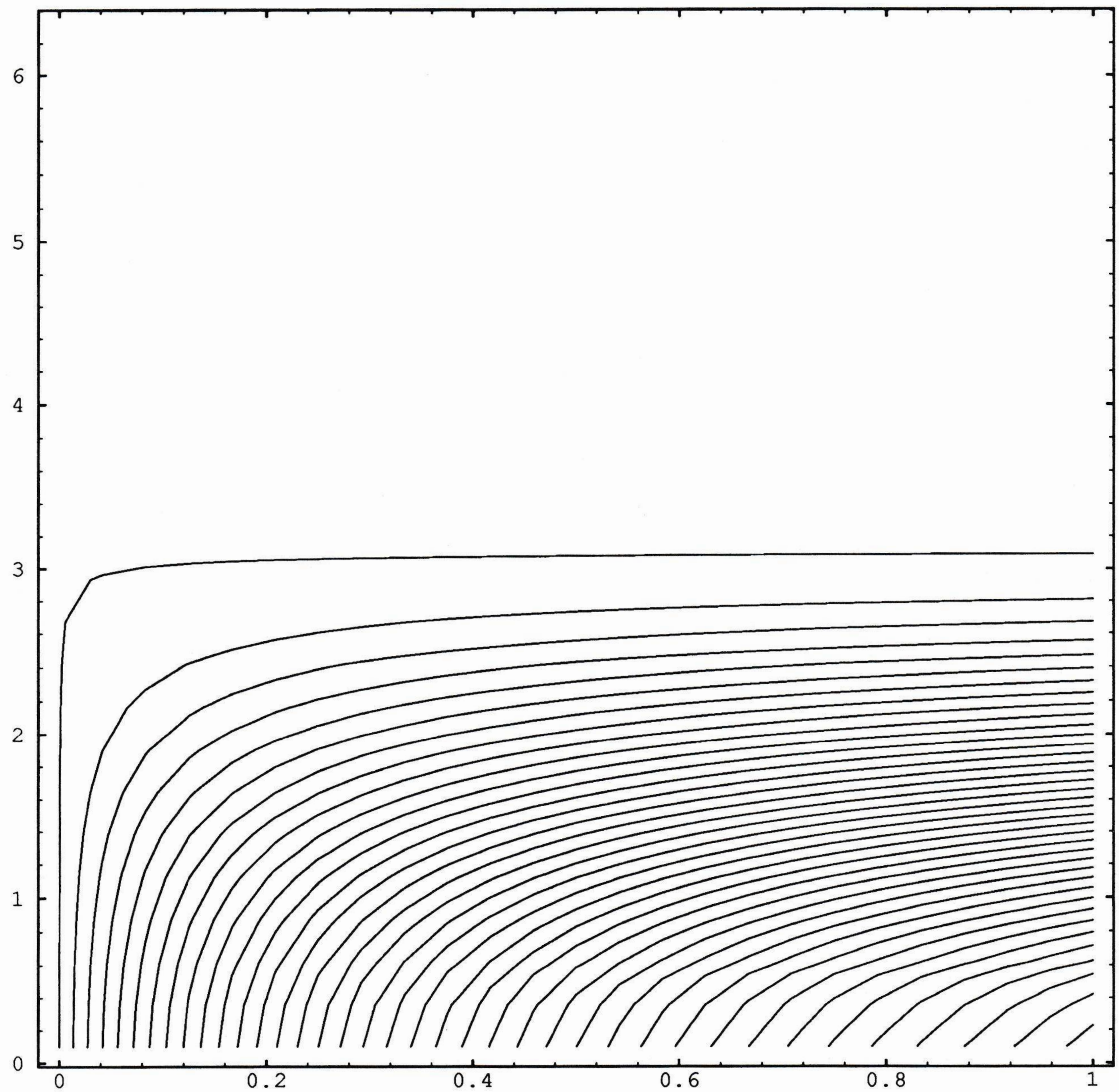
Plot3D[f[v,ρ], {v,0,1}, {ρ,0,2Pi}, AspectRatio->Automatic, Plotpoints->35]

3-8 Plot3D[f[v,ρ], {v, 0, 1}, {ρ, 0, 2π}]





Plotpoints -> 30   Contours -> 60



ContourPlot[f[v,p], {v, 0, 1}, {p, .1, 2Pi}, PlotPoints->25, Contours-> 40]