Salzman

Excellent job

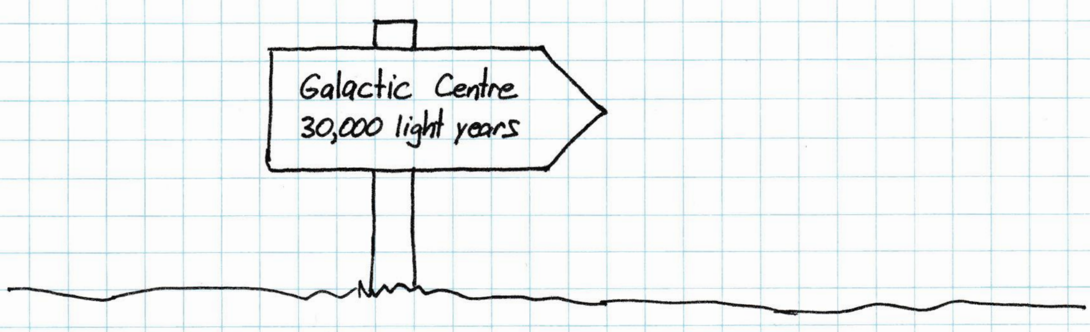
+

Problem 4

A rocket travels from Earth to the centre of the galaxy and returns. The rocket starts from rest, accelerates uniformly at 1-g till at the midpoint of the journey, then turns and decelerates uniformly at 1 g to rest at the galactic centre. Its return trip follows the same plan.

Find the round trip travel time (hint: 4 times the time for $\frac{1}{4}$ trip) as determined in the Earth reference frame and the proper time experienced by the astronauts.

Hint Use the Earth reference frame and a reference frame instantaneously at rest with respect to the rocket. Determine the change in the rocket's velocity wrt the instantaneous frame during a proper time increment $d\tau$. Use the transformation laws to obtain the corresponding Earth frame time interval dt and change in velocity wrt the Earth frame. Thus, obtain a differential equation for velocity as a function of time in the Earth frame. Integrate the equation to find position as a function of time and proceed to find the required results.



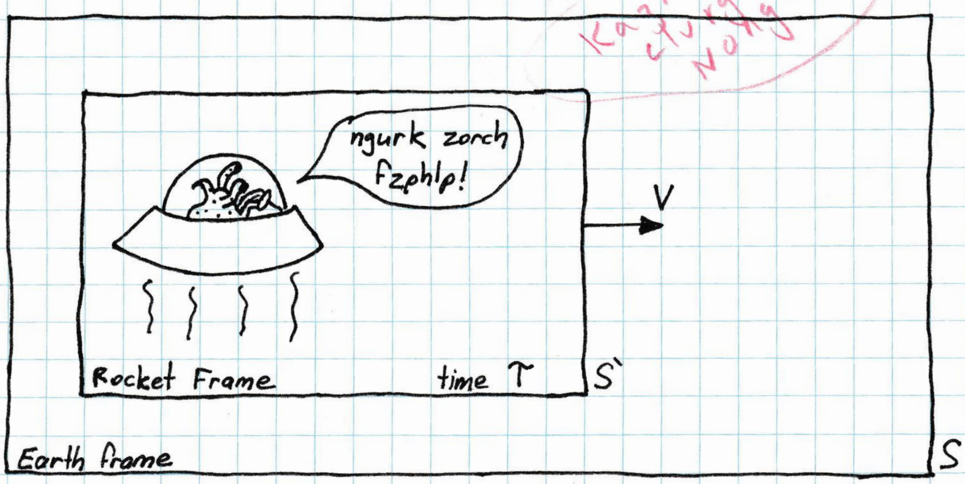
Now consider the rocket frame. We can't turn to SR for the whole story because the rocket is not inertial; Newton's 1st law does not hold in the rocket frame.

However, as Taylor + Wheeler point out, any frame will be inertial as long as you restrict yourself to the appropriate region in spacetime.

What is the appropriate region? We aren't concerned with massive objects — tidal gravity is not an issue here. The objection is that the rocket is accelerating, so after some amount of time the frame is no longer inertial.

The answer is to choose a frame that is spatially large but temporally small. Consider the setup below.

Kazing churgca Nolg mut!!



We will keep the rocket frame for an instant, after which it no longer is inertial so we'll choose another frame at a slightly higher speed which is inertial. (rather, in which the rocket is ~~inertial~~ at rest)

Suppose we're at time $t = \tau$. The rocket is instantaneously at rest in its reference frame which is travelling at speed V as measured by Terran observers.

The ansatz (sp?) here is that the rocket has an acceleration g in its momentary frame — [in other words, acceleration does not transform between different frames of reference. We did not discuss this point in class.]

transformation is not an issue, since the accel. is given as measured in the rocket's frame

~~Anyhow, suppose the rocket's acceleration is g in all frames. This being the case,~~ the change in the rocket's speed wrt its own frame in some proper differential amount of time is:

$$dV' = g\, d\tau$$

since the rocket remains at speeds ≪c wrt its inst. rest frame using a Newtonian result is fine in that frame

~~During~~ After this proper time interval $d\tau$, the rocket's speed as measured by Earthly observers will be:

X!clork
pronounced click

$$v + dv = \frac{dv' + v}{1 + v\,dv'}$$

$$= \frac{g\,d\tau + v}{1 + g v\,d\tau}$$

The denominator is in the form of 1 + something small. Applying the binomial theorem,

$$v + dv = [g\,d\tau + v]\left(1 + g v\,d\tau\right)^{-1}$$

$$= [g\,d\tau + v][1 - g v\,d\tau]$$

$$= g\,d\tau - \cancel{g^2 v (d\tau)^2} + v - g v^2 d\tau$$

$\to 0$ (second order differential)

So, the change in the rocket's speed wrt Earth (and consequently, the change in the new rocket frame's speed wrt the old rocket frame as measured by Earth) is:

$$dv = g(1 - v^2)\,d\tau$$

$$\frac{dv}{1 - v^2} = g\,d\tau$$

This is a differential equation relating the rocket's change of speed in Earth frame over a small time interval in the rocket frame. To get the rocket's speed at any time τ we sum all the dv's that occur for each $d\tau$ between 0 and τ. In other words, we integrate!

One of the best lessons I learned in junior mechanics is to tailor the integration method to suit the physical problem at hand. v ought not be sinusoidal on τ so my 1st reaction $v = \cos\theta$ is not the way to go. Partial fractions ought to work, but that will yield a logarithm which is ill-behaved close to the origin. I think a hyperbolic tangent substitution is the correct manuever here.

You know what's odd? I swear this was the same relationship between v and t for a projectile in air resistance $\propto v^2$. Kirengx

Could relativity be the effect of drag forces on the rocket ship due to the luminefirous ether? !

$$\int_0^{\tau} g\, d\tau^* = \int_0^{v} \frac{dv^*}{1 - v^{*2}}$$

$$v^* = \tanh \eta$$

$$dv^* = \operatorname{sech}^2\eta\, d\eta$$

$$g\tau = \int_0^{\tanh^{-1}(v)} \frac{\operatorname{sech}^2\eta}{1-\tanh^2\eta}\, d\eta$$

$$= \int_0^{\tanh^{-1}(v)} d\eta$$

$$= \tanh^{-1}(v)$$

Taking inverse of inverse TANH ☺ of both sides gives us the rocket speed as measured by Earth observers as a function of the rocket's proper time.

Nice

$$\boxed{v(\tau) = \tanh(g\tau)} \qquad (1)$$

We can integrate this function to find the rocket's position wrt Earth as a function of the rockets proper time. To do so, we need to explicitly write v's dependance on τ which is easily done via the chain rule.

$$v = \frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\frac{1}{\gamma} = \frac{dx}{d\tau}\sqrt{1-v^2}$$

So:

$$\frac{dx}{d\tau}\sqrt{1-v^2} = \tanh(g\tau)$$

$$dx = \frac{\tanh(g\tau)}{\sqrt{1-v^2}}\,d\tau$$

$$= \frac{\tanh(g\tau)}{\sqrt{1-\tanh^2(g\tau)}}\,d\tau$$

$$= \frac{\tanh(g\tau)}{\operatorname{sech}(g\tau)}\,d\tau$$

The differential equation relating the rocket's position and the rocket's proper time is:

$$dx = \sinh(g\tau)\,d\tau$$

Integrating both sides of the equation,

$$\int_0^x dx^* = \int_0^\tau \sinh(g\tau^*)\,d\tau^*$$

$$x = \frac{1}{g}\int_0^{g\tau} \sinh(u)\,du$$

$$= \frac{1}{g}\cosh(u)\Big|_0^{g\tau}$$

$$\boxed{x = \frac{1}{g}\left[\cosh(g\tau) - 1\right]} \qquad \text{(ii)}$$

Thanks goes to Richard G. for this next step. Note:

$$-\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2$$

$$= -\gamma^2 + \left(\frac{dx}{dt}\frac{dt}{d\tau}\right)^2$$

$$= -\gamma^2 + v^2\gamma^2$$

$$= \gamma^2(v^2-1)$$

$$= \frac{1-v^2}{v^2-1}$$

$$= -1$$

aren't 4-vectors cute

Using our expression for $x(\tau)$,

$$\frac{dx}{d\tau} = \sinh(g\tau)$$

And inserting it in the equation above, (after multiplying through by -1), we get:

$$+\left(\frac{dt}{d\tau}\right)^2 - \sinh^2(g\tau) = 1$$

Comparing this expression with the hyperbolic version of Pythagorus's Theorem $\cosh^2 z - \sinh^2 z = 1$ we can equate terms to find that:

$$\frac{dt}{d\tau} = \cosh(g\tau)$$

we chose the positive root of $\partial_\tau t$ because proper time intervals are the shortest time intervals between 2 events, so $\partial_\tau t > 1$.

From the last equation we have:

$$dt = \cosh(g\tau)\, d\tau$$

Integrating both sides:

$$\int_0^t dt^* = \int_0^\tau \cosh(g\tau^*)\, d\tau^*$$

$$\boxed{t = \frac{1}{g} \sinh(g\tau)} \qquad (iii)$$

Piecing a solution together

First, g needs to be put into more palatable units.

$$g = \frac{9.81\ m/s^2}{299792458 m/s} \left(\frac{\pi \times 10^7 s}{1\ yr} \right) = 1.028/yr$$

That worked out just swell! Now, according to Misner, Thorne and Wheeler, the distance to the centre of the galaxy is $X_g = 30$ k yr. Let t_g and T_g denote the roundtrip time according to Earth and rocket observers respectively.

We can use the inverse of eq (ii) to find the proper time for the round trip.

$$X = \frac{1}{g}\left[\cosh(g\tau) - 1\right] \implies \tau = \frac{1}{g}\cosh^{-1}(gX + 1)$$

For the quarter trip,

$$\frac{1}{4} T_g = \frac{1}{1.028/yr} \cosh^{-1}\left[(1.028/yr)(15\text{ kyr}) + 1\right]$$

$$= 10.0551\ yr$$

So, the round trip for people on the rocket will be:

$$\tau_g = 40.22 \text{ yr} = 40\text{yr} + 3\text{mo}$$

Now using equation (iii) we can calculate the time on Earth that passes during this round trip.

$$\frac{1}{4} t = \frac{1}{g} \text{SINH}\left(\frac{1}{4} g \tau\right)$$

$$t = \frac{4}{1.028/\text{yr}} \text{SINH}\left([1.028/\text{yr}][10.0551 \text{ yr}]\right)$$

$$= 60,000 \text{ yr} \qquad (60,004 \text{ yr})$$

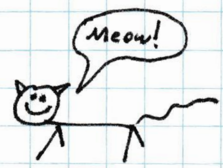
What is the non-relativistic answer?

$$t_{n.r.} = \sqrt{\frac{8 X_g}{g}} = \sqrt{\frac{8(2.8 \times 10^{20} \text{m})}{9.81 \text{ m/s}^2}}$$

$$= 15000000000 \text{ yr}$$

$$= 15 \text{ thousand million years}$$

which (I think) is about the age of the universe!



Recap

A rocket makes a round trip journey between Earth and the centre of the galaxy.

The astronauts age only 40 years during the trip while their friends and family age 60,000 years.

The only question left to answer is how much fuel does one need to carry in order to accelerate at 1g for 40 years?

At least you don't need to have 60,000 years worth of fuel!

Lary Niven used $\vec{B}$ fields to gather and fuse inter-stellar H.

Application of results of rocket calculation to the trip to the galactic center

(Note, in my old solution, $\tau \equiv ct$, rather than standing for proper time.)

So $X - X_0 = \frac{1}{2}$ distance to galactic center. The corresponding time interval is $\frac{1}{4}$ the interval for the entire trip, which consists of an acceleration and deceleration leg on both outbound and return trips.

Then: $\frac{a}{c^2}(x - x_0) = \frac{9.8\,m/s^2}{(3\times10^8\,m/s)^2}(10^4\,l.y) \simeq (3\times10^{-8}/s)(10^4 \cdot \pi\times10^7 s)$

$= 10^4 \gg 1$

$\Rightarrow \frac{a\tau}{c^2} \simeq \frac{a}{c^2}(x - x_0) \Rightarrow \tau = x - x_0 = 10^4\,l.y$

or $t = 10^4 y$.

So, the trip time is negligibly different from travel at the speed of light.

What is the proper time the astronauts experience? (Call it $\int ds$)

$$\int ds = \int \frac{dt}{\gamma} = \int \frac{dt}{1/\sqrt{1 - v^2/c^2}} = \int dt \sqrt{1 - \frac{(gt)^2/c^2}{1 + (gt/c)^2}}$$

$$= \int dt \left[1 + (gt/c)^2\right]^{-1/2} \qquad \text{Let } gt/c = \sinh\lambda$$

$$= \int \frac{c}{g} \frac{\cosh\lambda \, d\lambda}{\cosh\lambda} = \frac{c\lambda}{g}$$

So proper time $= \frac{g}{c}\sinh^{-1} gt/c = \frac{c}{g}\ln\left[\frac{gt}{c} + \sqrt{\left(\frac{gt}{c}\right)^2 + 1}\,\right]$

Application p2:

$$\simeq \frac{c}{g} \ln\left(2g\frac{t}{c}\right) = \frac{3\times10^{8}\, m/s}{9.8\, m/s^2} \ln\left[\frac{2\,(9.8\, m/s^2)(10^{4}\, y)(\pi\times10^{7}\, s/y)}{3\times10^{8}\, m/s}\right]$$

$$= (3.1\times10^{7}\, s) \ln\left[2\times10^{4}\right] = 31\times10^{7}\, s \simeq 10\, y$$

So, the round trip voyage would take the astronauts 40 y. // !!

(A solution using 4-Vectors)

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Phys 425.01
Due 3/6/96

Problem 4.

A rocket travels from Earth to the center of the galaxy and returns. The rocket starts from rest, accelerates uniformly at 1g till at the midpoint of the journey, then turns and decelerates uniformly to rest at the galactic center. Its return trip follows the same plan. Calculate the roundtrip travel time in the Earth reference frame.

$\Longrightarrow$

Let the Earth frame be represented by coordinates

(eqn1) $$(x_0, x_1, x_2, x_3) = (ct, x, y, z)$$

Assume the rocket motion is along the x-axis.

The four-velocity of the rocket in the earth frame is

(eqn2) $$(u_0, u_1, u_2, u_3) = \left(\frac{dx_0}{d\tau}, \frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau}\right)$$

$$= \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, 0, 0\right)$$

since motion is along x-axis

the four-acceleration of the rocket is

(eqn3) $$(a_0, a_1, a_2, a_3) = \left(\frac{du_0}{d\tau}, \frac{du_1}{d\tau}, \frac{du_2}{d\tau}, \frac{du_3}{d\tau}\right)$$

$$= \left(c\frac{d^2t}{d\tau^2}, \frac{d^2x}{d\tau^2}, 0, 0\right)$$

Note the following constraints on $\vec{u}$ and $\vec{a}$

<u>Normalization of $\vec{u}$:</u>

$$\vec{u}\cdot\vec{u} = u^{\mu}u_{\mu} = -(u_0)^2 + (u_1)^2 + (u_2)^2 + (u_3)^2$$

$$= -\left(\frac{dx_0}{d\tau}\right)^2 + \left(\frac{dx_1}{d\tau}\right)^2 + \left(\frac{dx_2}{d\tau}\right)^2 + \left(\frac{dx_3}{d\tau}\right)^2$$

$$= -\left(\frac{c\,dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 + \left(\frac{dz}{d\tau}\right)^2$$

$$= -\left(c\frac{dt}{dt}\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{dt}\frac{dt}{d\tau}\right)^2 + \left(\frac{dy}{dt}\frac{dt}{d\tau}\right)^2 + \left(\frac{dz}{dt}\frac{dt}{d\tau}\right)^2$$

$$= -(c\gamma)^2 + (V_x\gamma)^2 + (V_y\gamma)^2 + (V_z\gamma)^2$$

$$= \gamma^2\left(-c^2 + V_x^2 + V_y^2 + V_z^2\right)$$

$$= \gamma^2\left(-c^2 + V^2\right) = \frac{-c^2+V^2}{1-\left(\frac{V}{c}\right)^2} = \frac{c^2(-c^2+V^2)}{c^2 - V^2}$$

$$= -\frac{c^2(c-V^2)}{c-V^2} \Longrightarrow$$

(eqn4) $$\vec{u}\cdot\vec{u} = u^{\mu}u_{\mu} = -c^2$$

Orthogonality of $\vec{a}$ and $\vec{u}$:

(eqn 4) $$0 \underset{\uparrow}{=} \frac{d}{d\tau}\left(-\frac{1}{2}c^2\right) \underset{\uparrow}{=} \frac{d}{d\tau}\left(\frac{1}{2}\vec{u}\cdot\vec{u}\right)$$

since c is a constant

by 4

$$= \frac{1}{2}\left(\vec{u}\cdot\frac{d\vec{u}}{d\tau} + \vec{u}\cdot\frac{d\vec{u}}{d\tau}\right) = \vec{u}\cdot\vec{a}$$

Invariance of $\vec{a}$

(eqn 5) $$\vec{a}\cdot\vec{a} = a^\mu a_\mu = \text{proper acceleration}$$

Summarizing the constraints on $\vec{u}$ and $\vec{a}$ and putting in terms of the problem at hand gives

(eqn 6) $$-(u_0)^2 + (u_1)^2 = -c^2$$

(eqn 7) $$-u_0 a_0 + u_1 a_1 = 0$$

(eqn 8) $$g^2 \underset{\uparrow}{=} -(a_0)^2 + (a_1)^2$$

since we are given that the proper acceleration is g solving (7) for a_0 gives

(eqn 9) $$a_0 = \frac{u_1 a_1}{u_0}$$

using (9) in (8) gives

(eqn 10) $$g^2 = -\left(\frac{u_1 a_1}{u_0}\right)^2 + (a_1)^2 = (a_1)^2\left[1-\left(\frac{u_1}{u_0}\right)^2\right]$$

Now, solve (6) for a_1

(eqn 11) $$a_1 = \frac{g}{\sqrt{1-\left(\frac{u_1}{u_0}\right)^2}} = \frac{-g}{\sqrt{\left(\frac{u_1}{u_0}\right)^2 - 1}}$$

substitute (11) in (9)

$$a_0 = \left(\frac{u_1}{u_0}\right)\left(\frac{-g}{\sqrt{\left(\frac{u_1}{u_0}\right)^2-1}}\right) = \frac{-g u_1}{\sqrt{-(u_0)^2 + (u_1)^2}}$$

$$\underset{\text{by 7}}{=} \frac{-g u_1}{\sqrt{-c^2}} = \frac{g u_1}{\sqrt{c^2}} \Longrightarrow$$

(eqn12) $$\boxed{a_0 = \frac{g u_1}{c}}$$

Solving (7) for a_1 gives

$$a_1 = \frac{u_0 a_0}{u_1} \underset{\text{by 12}}{=} \frac{u_0}{u_1}\left(\frac{g u_1}{c}\right) \Longrightarrow$$

(eqn13) $$\boxed{a_1 = \frac{g u_0}{c}}$$

Note that

$$\frac{d^2 u_1}{d\tau^2} = \frac{d}{d\tau}\frac{du_1}{d\tau} = \frac{da_1}{d\tau} \underset{\text{by 13}}{=} \frac{d}{d\tau}\left(\frac{g u_0}{c}\right)$$

$$= \frac{g}{c}\frac{du_0}{d\tau} = \frac{g a_0}{c} = \frac{g}{c}\left(\frac{g u_1}{c}\right) \Longrightarrow$$

(eqn14) $$\boxed{\frac{d^2 u_1}{d\tau^2} - \frac{g^2}{c^2} u_1 = 0}$$

Now, solve the the above differential eqn:

$$r^2 + 0r - \frac{g^2}{c^2} = 0 \implies$$

$$r = \pm\sqrt{-4(1)\left(\frac{-g^2}{c^2}\right)}\Big/2 = \pm\frac{g}{c} \implies$$

(eqn15) $$u_1 = Ae^{g\tau/c} + Be^{-g\tau/c}$$

To find A & B, we make use of the initial conditions:

(eqn16) $u_1(\tau=0) = 0$ (Rocket starts from rest)

(eqn17) $\left.\frac{du_1}{d\tau}\right|_{\tau=0} = g$ (Rocket starts at the earth where acceleration is g)

Using (16) in (15) gives

$$0 = A + B \implies$$

(eqn18) $B = -A$

Using (18) in (15) gives

(eqn 19) $$u_1 = A\left(e^{g\tau/c} - e^{-g\tau/c}\right)$$

Differentiating (19) w.r.t. τ gives

$$\frac{du_1}{d\tau} = A\left(\frac{g}{c}e^{g\tau/c} + \frac{g}{c}e^{-g\tau/c}\right) \Longrightarrow$$

(eqn 20) $$\frac{du_1}{d\tau} = \frac{Ag}{c}\left(e^{g\tau/c} + e^{-g\tau/c}\right)$$

Using (17) in (20) gives

$$g = \frac{Ag}{c}\left(1+1\right) = \frac{2Ag}{c} \Longrightarrow$$

$$\frac{2A}{c} = 1 \Longrightarrow$$

(eqn 21) $$A = \frac{c}{2}$$

Using (21) in (19) gives

(eqn 22) $$\boxed{u_1 = \frac{dx}{d\tau} = c\left(\frac{e^{g\tau/c} - e^{-g\tau/c}}{2}\right) = c\sinh\left(\frac{g\tau}{c}\right)}$$

Using (22) in (6) gives

$$-(u_0)^2 + c^2 \sinh^2\left(\frac{g\tau}{c}\right) = -c^2 \Rightarrow$$ Divide by $-c^2$

(eqn23) $$\frac{1}{c^2}(u_0)^2 - \sinh^2\left(\frac{g\tau}{c}\right) = 1$$

Recall the identity

(eqn24) $$\cosh^2\theta - \sinh^2\theta = 1$$

Thus, we see that

$$\frac{1}{c^2}(u_0)^2 = \cosh^2\left(\frac{g\tau}{c}\right) \Longrightarrow$$

(eqn25) $$\boxed{u_0 = \frac{dx_0}{d\tau} = \frac{c\,dt}{d\tau} = c\cosh\left(\frac{g\tau}{c}\right)}$$

where we have chosen the positive root since

(eqn26) $$u_0 = \frac{dx_0}{d\tau} = c\frac{dt}{d\tau} = c\gamma > 1$$

Integrating (22) gives

(eqn27) $$X = c\int \sinh\left(\frac{g\tau}{c}\right) = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right) + K$$

where K is a constant of integration.

To determine K, we use the initial condition

(eqn28) $$X(\tau=0) = 0$$ (Rocket starts at origin of Earth frame)

using (28) in (27) gives

$$0 = \frac{c^2}{g}\cosh(0) + K = \frac{c^2}{g} + K \implies$$

(eqn29) $$K = -c^2/g$$

using (29) in (28) gives

(eqn30) $$X = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau}{c}\right) - 1\right]$$

Integrating (25) gives

(eqn36) $g \simeq 1.03280 \frac{c}{y}$

We can use eqn (30) to determine the proper time τ_{half} to travel halfway to the galactic center (and thus the proper time required for the round-trip by multiplying this result by four). First, we solve (30) for τ:

$$\frac{gx}{c^2} = \cosh\left(\frac{g\tau}{c}\right) - 1 \implies$$

$$\cosh\left(\frac{g\tau}{c}\right) = \frac{gx}{c^2} + 1 \implies$$

$$\frac{g}{c}\tau = \operatorname{arccosh}\left(\frac{gx}{c^2} + 1\right) \implies$$

(eqn37) $$\boxed{\tau = \frac{c}{g}\operatorname{arccosh}\left(\frac{gx}{c^2} + 1\right)}$$

Traveling halfway to the galactic center corresponds to $\frac{1}{2} x_G$ as defined in eqn (35).

Thus

(eqn38) $$\tau_{half} = \frac{c}{g} \operatorname{arccosh}\left[\frac{g\left(\frac{1}{2} x_G\right)}{c^2} + 1\right]$$

substituting in values from (35) & (36) gives

$$\tau_{half} = \left(\frac{c}{1.0328\ c/y}\right) \operatorname{arccosh}\left[\frac{1.0328\ \frac{c}{y}\left(\frac{1}{2} \cdot 30{,}000\, c \cdot y\right)}{c^2} + 1\right] \Longrightarrow$$

(eqn39) $$\boxed{\tau_{half} \simeq 10.0128687 \text{ years} \simeq 10.013 \text{ years}}$$

The proper time required for the roundtrip, $\tau_{roundtrip}$, is 4 times τ_{half}:

(eqn40) $$\boxed{\tau_{roundtrip} = 4 \cdot \tau_{half} \simeq 40.05 \text{ years}}$$

Now, we can use eqns (34) & (38) to compute the time required in the Earth frame to travel halfway to the galactic center (denoted t_{half})

(eqn 41) $$\boxed{t_{half} = \frac{c}{g} \sinh\left(\frac{g\tau_{half}}{c}\right)}$$

substituting in values from (39) & (36) gives

$$t_{half} = \left(\frac{c}{1.0328\ c/y}\right) \sinh\left[\frac{(1.0328\ c/y)(10.0128687\ y)}{c}\right]$$

$\Longrightarrow$

(eqn 42) $$\boxed{t_{half} \simeq 15{,}000.9686 \text{ years} \simeq 15{,}001 \text{ years}}$$

The time required for the roundtrip in the Earth frame, $t_{roundtrip}$ is 4 times t_{half}:

(eqn 43) $$\boxed{t_{roundtrip} = 4 \cdot t_{half} \simeq 60{,}003.87 \text{ years} \simeq 60{,}004 \text{ years}}$$