PJS

100

final exam

thurs dec 16

8 - 10:30 am

hw

303/1,5

~~333/10,11,12~~ → 344/15,17,19,20

hw

do last semesters final test

MA 245.02 EXAM # 3 F 93

(TAKE HOME: DUE ~~THURS~~ tues. NOV. ~~18~~ 23, 1993, 8:10 A,M.)

1. $x^2y'' + 7xy' + (8+x^2)y = 0$

(a) Find the indicial equation.

(b) Find the first 5 nonzero terms of y_1 , a series solution about $x=0$ corresponding to the larger root of the indicial equation.

(c) Using Bessel functions, express y_1 in closed form.

(d) Find a linearly independent solution y_2. (Use a formula which involves an integral. Simplify the integral, but don't try to evaluate it.)

2. $x^2y'' + 4xy' + (2-x^2)y = 0$

(a) Find the indicial equation.

(b) Find the first 5 nonzero terms of y_1, a series solution about $x=0$ corresponding to the larger root of the indicial equation.

(c) Express y_1 in closed form. (Consult class notes or the series chapter of a calculus text.)

(d) Find a linearly independent solution, y_2 , in closed form. (This will involve the evaluation of an integral.)

Before handing in your solutions, please sign the statement below:

I have not requested assistance from nor offered assistance to anyone. The work which I am handing in represents only my own efforts.

Peter Jay Salzman

(your signature)

① $x^2y'' + 7xy' + (8+x^2)y = 0$

a) Find indicial equation

By inspection:

$p(x) = \frac{7}{x}$ $\quad$ $a(x) = xp(x) = 7$ $\quad$ $a_0 = 7$

$q(x) = \frac{(8+x^2)}{x^2}$ $\quad$ $b(x) = x^2p(x) = 8 + x^2$ $\quad$ $b_0 = 8$

$r^2 + (7-1) + 8 = r^2 + 6r + 8 = (r+4)(r+2)$

$\Longrightarrow r = -4, -2$

b) Assume a series solution in the form of:

$$y = \sum_{n=0}^{\infty} a_n x^{n-2}$$

$$y' = \sum_{n=0}^{\infty} a_n (n-2) x^{n-3}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n-2)(n-3) x^{n-4}$$

Plug assumed solutions into original equation:

$$X^2\sum_{n=0}^{\infty}(n-2)(n-3)a_n X^{n-4} + 7X\sum_{n=0}^{\infty}(n-2)a_n X^{n-3} + (8+X^2)\sum_{n=0}^{\infty}a_n X^{n-2} = 0$$

Bring factors into infinite summations; distribute 3rd factor

$$\sum_{n=0}^{\infty}(n-2)(n-3)a_n X^{n-2} + \sum_{n=0}^{\infty}(n-2)7a_n X^{n-2} + \sum_{n=0}^{\infty}8a_n X^{n-2} + \sum_{n=0}^{\infty}a_n X^{n} = 0$$

Combine 1st 3 summations

$$\sum_{n=0}^{\infty}\Big[(n-2)(n-3)a_n + 7(n-2)a_n + 8a_n\Big]X^{n-2} + \sum_{n=0}^{\infty}a_n X^{n} = 0$$

Simplify:

$$(n-2)(n-3)a_n + 7na_n - 14a_n + 8a_n$$

$$a_n(n^2-5n+6) + 7na_n - 14a_n + 8a_n$$

$$a_n n^2 - 5na_n + 6a_n + 7na_n - 14a_n + 8a_n$$

$$a_n n^2 + 2na_n = a_n(n^2+2n) = a_n n(n+2)$$

Rewrite last equation; change last term's index:

$$\sum_{n=0}^{\infty}\Big[a_n n(n+2)\Big]X^{n-2} + \sum_{n=2}^{\infty}a_{n-2}X^{n-2} = 0$$

There is no contribution of the 1st summation at $n=0$. Evaluate 1st summation at $n=1$.

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$$\sum_{n=2}^{\infty}\left(a_n n(n+2)\right) x^{n-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n-2} + \frac{3a_1}{x} = 0$$

Combine both infinite summations

$$\sum_{n=2}^{\infty}\left[a_n n(n+2) + a_{n-2}\right] x^{n-2} + \frac{3a_1}{x} = 0$$

This is true iff

a) $a_1 = 0$

b) $a_n = -\dfrac{a_{n-2}}{n(n+2)}$

The 2nd constraint is the recurrence relationship. The 1st constraint along with the 2nd constraint says that all $a_{odd} = 0$. Thus, for even values of n, let $n = 2m$; then:

$$a_{2m} = -\frac{a_{2m-2}}{(2m+2)(2m)} = -\frac{a_{2m-2}}{2^2(m+1)m} \qquad m = 1, 2, 3, \text{etc}$$

$m=1$: $\quad a_2 = -\dfrac{a_0}{4(2)(1)} = -\dfrac{a_0}{8}$

$m=2$: $\quad a_4 = -\dfrac{a_2}{4(3)(2)} = \dfrac{a_0}{2^4\, 3!\, 2!} = \dfrac{a_0}{192}$

$m=3$: $\quad a_6 = -\dfrac{a_4}{4(4)(3)} = -\dfrac{a_0}{2^6\, 4!\, 3!} = -\dfrac{a_0}{9216}$

$m=5$: $\quad a_8 = -\dfrac{a_6}{4(6)(5)} = \dfrac{a_0}{2^8\, 5!\, 4!} = \dfrac{a_0}{737280}$

In general, $a_{2m} = \dfrac{(-1)^m a_0}{2^{2m}(m+1)!\,m!}$ $\qquad m = 1, 2, 3, \ldots$

The solution, $y_1(x)$ will be in the form

$$y_1(x) = a_0 x^{0-2} + a_1 x^{1-2} + a_2 x^{2-2} + a_3 x^{3-2} + a_4 x^{4-2} + a_5 x^{5-2} + \ldots$$

As I pointed out, $a_{odd} = 0$, so:

$$y_1(x) = a_0 x^{-2} + a_2 + a_4 x^2 + a_6 x^4 + a_8 x^6 + a_{10} x^8 + \ldots$$

$$y_1(x) = \frac{a_0}{x^2} - \frac{a_0}{8} + \frac{a_0}{192} x^2 - \frac{a_0}{9216} x^4 + \frac{a_0}{737280} x^6 - \ldots + \ldots$$

$$y_1(x) = a_0\left(\frac{1}{x^2} - \frac{1}{8} + \frac{x^2}{192} - \frac{x^4}{9216} + \frac{x^6}{737280} - \ldots + \ldots\right)$$

$$y_1(x) = \frac{a_0}{x^2} + \sum_{n=1}^{\infty} \frac{(-1)^n a_0}{2^{2n}(n+1)!\,n!} x^{2n-2}$$

$$= a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-2}}{2^{2n}(n+1)!\,n!}$$

Looks like $J_1(t) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}t\right)^{2n+1}}{(n+1)!\,n!}$

I will try to express my solution $a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-2}}{2^{2n}(n+1)!n!}$ in terms of

$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{2n+1}}{(n+1)!\,n!}$.

$$a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-2}}{2^{2n}(n+1)!n!} = a_0 x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n+1)!n!} = a_0 x^{-2} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{2n}}{(n+1)!\,n!}$$

$$= a_0 x^{-3} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}x)^{2n+1}}{(n+1)!\,n!} = a_0 x^{-3} J_1(x)$$

$$y_1 = a_0 x^{-3} J_1(x)$$

$$y_2 = a_0 x^{-3} J_1(x) \int \left(a_0 x^{-3} J_1(x)\right)^{-2} e^{-\int \frac{7}{x} dx} \, dx$$

$$= \frac{a_0^2 J_1(x)}{x^3} \int \frac{x^6}{J_1^2(x)} e^{\ln x^{-7}} dx$$

$\frac{5}{5}$

$$= \frac{a_0^2 J_1(x)}{x^3} \int \frac{x^6 x^{-7}}{J_1^2(x)} dx = \frac{a_0^2 J_1(x)}{x^3} \int \frac{dx}{x J_1^2(x)}$$

② $x^2y'' + 4xy' + (2-x^2)y = 0$

a) Find indicial equation

By inspection:

$p(x) = \frac{4x}{x^2} = \frac{4}{x}$ $\qquad a(x) = xp(x) = 4$

$q(x) = \frac{(2-x^2)}{x^2}$ $\qquad b(x) = x^2q(x) = 2-x^2$

$r^2 + (4-1)r + 2 = r^3 + 3r + 2 = (r+2)(r+1)$

$\Rightarrow r = -1, -2$

b) Assume a series solution in the form of:

$$y = \sum_{n=0}^{\infty} a_n x^{n-1}$$

$$y' = \sum_{n=0}^{\infty} a_n (n-1) x^{n-2}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n-1)(n-2) x^{n-3}$$

Plug assumed solution into original equation.

$$X^2 \sum_{n=0}^{\infty} a_n(n-1)(n-2)X^{n-3} + 4X \sum_{n=0}^{\infty} a_n(n-1)X^{n-2} + (2-X^2)\sum_{n=0}^{\infty} a_n X^{n-1} = 0$$

Bring all factors into summation. Distribute last factor.

$$\sum_{n=0}^{\infty} a_n(n-1)(n-2)X^{n-1} + \sum_{n=0}^{\infty} 4a_n(n-1)X^{n-1} + \sum_{n=0}^{\infty} 2a_n X^{n-1} - \sum_{n=0}^{\infty} a_n X^{n+1} = 0$$

Combine 1st 3 infinite summations.

$$\sum_{n=0}^{\infty} \left[a_n(n-1)(n-2) + 4a_n(n-1) + 2a_n \right] X^{n-1} - \sum_{n=0}^{\infty} a_n X^{n+1} = 0$$

Simplify: $a_n(n^2 - 3n + 2) + 4a_n n - 4a_n + 2a_n$

$$a_n n^2 - 3a_n n + 2a_n + 4a_n n - 4a_n + 2a_n$$

$$a_n n^2 + a_n n = a_n(n^2 + n)$$

Rewrite last equation

$$\sum_{n=0}^{\infty} \left[a_n(n^2+n) \right] X^{n-1} - \sum_{n=0}^{\infty} a_n X^{n+1}$$

Change index of 2nd summation. Start 1st summation from $n=1$ (no contribution at $n=0$.

$$\sum_{n=1}^{\infty} \left[a_n(n^2+n) \right] X^{n-1} - \sum_{n=2}^{\infty} a_{n-2} X^{n-1}$$

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Evaluate the 1st sum at $n=1$

$$\sum_{n=2}^{\infty}\left[a_n(n^2+n)\right]X^{n-1} - \sum_{n=2}^{\infty} a_{n-2}X^{n-1} + 2a_1 = 0$$

Combine all

$$\sum_{n=2}^{\infty}\left[a_n(n^2+n) - a_{n-2}\right]X^{n-1} + 2a_1 = 0$$

Which is true iff:

1) $a_1 = 0$

2) $a_n(n^2+n) - a_{n-2} = 0$ for $n \geq 2$

<u>recurrence formula:</u>

$$a_n = \frac{a_{n-2}}{n^2+n} \qquad n \geq 0$$

note that $a_{odd} = 0$.

Determination Of Coefficients

$\underline{n=2}$: $a_2 = \dfrac{a_0}{3\cdot 2} = \dfrac{a_0}{3!}$

$\underline{n=4}$: $a_4 = \dfrac{a_2}{5\cdot 4} = \dfrac{a_0}{5!}$

$\underline{n=6}$: $a_6 = \dfrac{a_4}{7\cdot 6} = \dfrac{a_0}{7!}$

$\underline{n=8}$: $a_8 = \dfrac{a_6}{9\cdot 8} = \dfrac{a_0}{9!}$

$\underline{n=10}$: $a_{10} = \dfrac{a_8}{11\cdot 10} = \dfrac{a_0}{11!}$

The solution $Y_1(x)$ is in the form of:

$$Y_1(x) = a_0 X^{0-1} + a_1 X^{1-1} + a_2 X^{2-1} + a_3 X^{3-1} + a_4 X^{4-1} + a_5 X^{5-1} + a_6 X^{6-1} + \dots$$

Since all $a_{odd} = 0$,

$$Y_1(x) = a_0 X^{-1} + a_2 X^{1} + a_4 X^{3} + a_6 X^{5} + a_8 X^{7} + a_{10} X^{9} + \dots$$

$$= \frac{a_0}{X} + \frac{a_0}{3!} X + \frac{a_0}{5!} X^3 + \frac{a_0}{7!} X^5 + \frac{a_0}{9!} X^7 + \frac{a_0}{11!} X^9 + \dots$$

Thus,

$$Y_1(x) = a_0 \left[\frac{1}{x} + \frac{x}{3!} + \frac{x^3}{5!} + \frac{x^5}{7!} + \frac{x^7}{9!} + \frac{x^9}{11!} + \dots\dots \right]$$

If we multiply both sides by x^2,

$$x^2 Y_1(x) = a_0 \left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \frac{x^9}{9!} + \frac{x^{11}}{11!} + \dots \right]$$

The series on the right is the Taylor polynomial expansion series for SINHX (obtained by "Calculus" by Steward).

$$x^2 Y_1(x) = a_0 \, \text{SINH}\, x$$

Finally, the solution:

$$Y_1(x) = \frac{a_0 \, \text{SINH}\, x}{x^2}$$

We can find $Y_2(x)$ using $Y_1(x)$:

$$Y_2(x) = \frac{a_0 \, \text{SINH}\, x}{x^2} \int \frac{x^4}{\text{SINH}^2 x} e^{-\int \frac{4}{x} dx} dx = \frac{a_0 \, \text{SINH}\, x}{x^2} \int \frac{dx}{\text{SINH}^2 x}$$

The integral is easily solved if we remember that

$$\frac{d}{dx}\left[\text{COTH}\, x \right] = -\text{CSCH}^2 x.$$

$$Y_2(x) = \frac{a_0 \, \text{SINH}\, x}{x^2} \int \frac{dx}{\text{SINH}^2 x} = \frac{a_0 \, \text{SINH}\, x}{x^2} \int \text{CSCH}^2 x \, dx$$

$$Y_2(x) = \frac{a_0 \sinh x}{x^2}\left(-\coth x\right) = -\frac{a_0 \sinh x \cosh x}{x^2 \sinh x}$$

$$= -\frac{a_0 \cosh x}{x^2}$$

(note a_0 can be + or −)

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CHECK: $\quad Y_1(x) = \dfrac{a_0 \sinh x}{x^2}$

$y = a_0 x^{-2} \sinh x \qquad y' = \dfrac{a_0 \cosh x}{x^2} - \dfrac{a_0 2 \sinh x}{x^3}$

$$y'' = \frac{a_0 6 \sinh x}{x^4} - \frac{a_0 4 \cosh x}{x^3} + \frac{a_0 \sinh x}{x^2}$$

$$x^2 y'' + 4xy' + (2 - x^2) y = 0$$

$$x^2\left[\frac{6 \sinh x}{x^4} - \frac{4 \cosh x}{x^3} + \frac{\sinh x}{x^2}\right] + 4x\left[\frac{\cosh x}{x^2} - \frac{2 \sinh x}{x^3}\right] + (2 - x^2)\left[\frac{\sinh x}{x^2}\right] \stackrel{?}{=} 0$$

(note: I divided by a_0)

$$\cancel{\frac{6 \sinh x}{x^2}} - \cancel{\frac{4 \cosh x}{x}} + \cancel{\sinh x} + \cancel{\frac{4 \cosh x}{x}} - \cancel{\frac{8 \sinh x}{x^2}} + \cancel{\frac{2 \sinh x}{x^2}} - \cancel{\sinh x} \stackrel{?}{=} 0$$

checks out.